

Analysis

Demarri Dosunmu

September 8, 2026

Part I Real

1 Fundamental Theorem of Algebra

1.1 Axler's proof

Source: [Axl24, p. 125].

Theorem 1.1 (8.8). *For $a_0, \dots, a_n$ complex numbers, $n \geq 1$, $a_n \neq 0$, and $P(z) = \sum_{k=0}^n a_k z^k$,*

$$P(z) = 0 \quad \text{for some } z \in \mathbb{C}.$$

PROOF. WLOG assume $a_n = 1$. Let

$$\mu = \inf |P(z)| \quad (z \in \mathbb{C}).$$

For $R = |z|$,

$$|P(z)| \geq R^n [1 - |a_{n-1}|R^{-1} - \dots - |a_0|R^{-n}],$$

which we obtain by dividing the polynomial by z^n , taking the modulus of both sides, and flipping signs in modulus distribution, preserving equality/ordering. The right side of the above tends to ∞ as $R \rightarrow \infty$. Hence there exists R_0 such that

$$|z| > R_0 \implies |P(z)| > \mu.$$

Since $|P|$ is continuous on the closed ball at 0 of radius R_0 in z , $|P(z_0)| = \mu$ must be attained for some z_0 by uniform continuity. Suppose $\mu \neq 0$. Let

$$Q(z) = \frac{P(z + z_0)}{P(z_0)}.$$

Then Q is a nonconstant polynomial, $Q(0) = 1$, and $|Q(z)| \geq 1$ for all z . Write

$$Q(z) = 1 + b_k z^k + \dots + b_n z^n, \quad b_k \neq 0.$$

Let θ be such that $e^{ik\theta} = -1$. Then for $r \in \mathbb{R}_+$ and $r^k |b_k| < 1$,

$$\left| 1 + b_k r^k e^{ik\theta} \right| = 1 - r^k |b_k|,$$

hence

$$|Q(re^{i\theta})| \leq 1 - r^k [|b_k| - r |b_{k+1}| - \dots - r^{n-k} |b_n|].$$

But as $r \rightarrow 0$, the expression in brackets is greater than 0, implying $|Q(re^{i\theta})| < 1$, a contradiction. Hence $\mu = 0$, and consequently, $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$. $\square$

1.2 Rudin's proof

Source: [Rud76, pp. 184–185].

PROOF. For p a nonconstant polynomial with complex coefficients and highest-order nonzero term $c_m z^m$, $|p(z)|/|z|^m \rightarrow |c_m|$ as $|z| \rightarrow \infty$ implies $|p(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$. Thus the continuous function $z \mapsto |p(z)|$ has a global minimum at some point $\zeta \in \mathbb{C}$. Suppose $p(\zeta) \neq 0$. Define

$$q(z) = \frac{p(z + \zeta)}{p(\zeta)}.$$

The function $z \mapsto |q(z)|$ then has a global minimum value of 1 at $z = 0$. Write

$$q(z) = 1 + a_k z^k + \cdots + a_m z^m,$$

where k is the smallest positive integer such that the coefficient of z^k is nonzero (i.e., $a_k \neq 0$). Let $\beta \in \mathbb{C}$ be such that

$$\beta^k = -\frac{1}{a_k}.$$

There exists $c > 1$ such that

$$t \in (0, 1) \implies |q(t\beta)| \leq |1 + a_k t^k \beta^k| + t^{k+1} c = 1 - t^k(1 - tc).$$

Hence for $t = \frac{1}{2c}$,

$$|q(t\beta)| < 1,$$

contradicting the assumption that the global minimum of $z \mapsto |q(z)|$ is 1, implying $p(\zeta) = 0$. $\square$

2 Simplexes and Chains

A function $\mathbf{f} : V \rightarrow W$ between vector spaces is **affine** if $\mathbf{f} - \mathbf{f}(\mathbf{0})$ is linear (i.e., translate of a homomorphism).

We define the **standard simplex** by

$$Q^k = \left\{ \mathbf{u} \mid \mathbf{u} = \sum_{i=1}^k \alpha_i \mathbf{e}_i, \alpha_i \geq 0, \sum \alpha_i \leq 1 \right\}$$

with the standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^k$. The **oriented k -simplex**

$$\sigma = [\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k], \quad \mathbf{x}_i \in Q^k$$

is the k -surface in $\mathbb{R}^n$ given by affine mapping

$$\sigma \left(\sum_{i=1}^k \alpha_i \mathbf{e}_i \right) = \mathbf{x}_0 + \sum_{i=1}^k \alpha_i (\mathbf{x}_i - \mathbf{x}_0).$$

An **affine k -chain** Γ in an open set $E \subset \mathbb{R}^n$ is a collection of finitely many oriented affine k -simplexes $\sigma_1, \dots, \sigma_r$ in E . Then for a k -form ω in E ,

$$\int_{\Gamma} \omega = \sum_{i=1}^r \int_{\sigma_i} \omega.$$

The **boundary** of an oriented affine k -simplex σ is the affine $(k-1)$ -chain

$$\partial\sigma = \sum_{j=0}^k (-1)^j [\mathbf{x}_0, \dots, \mathbf{x}_{j-1}, \mathbf{x}_{j+1}, \dots, \mathbf{x}_k]$$

(for each element, consider the simplex without the element, and sum these modified simplexes with alternating sign; see alternating multilinear forms, sequence of minor determinants). Partitioning sets into simplexes is called **triangulation**. Any n -cell can be triangulated and hence regarded as a chain.

For $T : E \rightarrow V$ a C^m function, $E \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ open, and σ an oriented affine k -simplex in E , the composite function $\Phi = T \circ \sigma$, a k -surface in V , is an **oriented k -simplex of class C^m** .

A finite collection of oriented k -simplexes $\Psi = (\Phi_1, \dots, \Phi_r)$ of class C^2 in V is a **k -chain of class C^2** in V , and for a k -form ω in V ,

$$\int_{\Psi} \omega = \sum_{i=1}^r \int_{\Phi_i} \omega$$

for $\Psi = \sum \Phi_i$. Further, for an affine chain $\Gamma = \sum \sigma_i$ and $\Phi_i = T \circ \sigma_i$,

$$\Psi = T \circ \Gamma$$

(i.e., $T(\sum \sigma_i) = \sum T\sigma_i$).

The **boundary** $\partial\Phi$ of the oriented k -simplex $\Phi = T \circ \sigma$ is the $(k-1)$ chain

$$\partial\Phi = T(\partial\sigma),$$

and the boundary $\partial\Psi$ of the k -chain $\Psi = \sum \Phi_i$ is the $(k-1)$ chain

$$\partial\Psi = \sum \partial\Phi_i.$$

Let σ_I denote the identity mapping of domain Q^n , thence a positively-oriented n -simplex in $\mathbb{R}^n$. For $T : Q^n \rightarrow E \subset \mathbb{R}^n$ invertible and C^2 with positive Jacobian for the interior of Q^n , if $E = T(Q^n)$, then the **positively-oriented boundary** of the set E is the $(n-1)$ -chain

$$\partial T = T(\partial\sigma_I),$$

and denoted by ∂E .

3 Stokes' Theorem

For Ψ a k -chain of class C^2 in $V \subset \mathbb{R}^m$ open and ω a C^1 $(k-1)$ -form,

$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega.$$

(**Fundamental quale:** changes by ω in the interior of a set determine uniquely what ω accumulates on the border of that set.)

4 Closed and Exact

For a k -form ω in $E \subset \mathbb{R}^n$ open, if λ is a $(k-1)$ -form in E with

$$\omega = d\lambda,$$

then ω is **exact** in E (note this involves the C^2 condition; it is the canonical higher form of a form which already exists in E , so no information is gained by taking the exterior derivative).

For a C^1 k -form ω in $E \subset \mathbb{R}^n$ open with

$$d\omega = 0,$$

ω is **closed**. Hence all exact forms are closed.

4.1 More Theorems

Theorem 4.1 (10.38). *For $E \subset \mathbb{R}^n$ convex and open, $F \in C^1(E)$, $p \in \mathbb{Z}_+$, $p \leq n$, and $(D_j F)(\mathbf{x}) = 0$ for $(p < j \leq n)$, there exists a C^1 function $f : E \rightarrow \mathbb{R}$ with*

$$\left(\frac{\partial F}{\partial x_p} \right) (\mathbf{x}) = f(\mathbf{x}), \quad \left(\frac{\partial f}{\partial x_j} \right) (\mathbf{x}) = 0 \quad (p < j \leq n).$$

Theorem 4.2 (10.39, Poincaré's Lemma). *For $E \subset \mathbb{R}^n$ convex and open, $k \in \mathbb{Z}_+$, ω a closed k -form of class C^1 in E , there exists $(k-1)$ -form λ in E with $\omega = d\lambda$ (i.e., closed forms are exact in convex sets).*

Theorem 4.3 (10.40). *For $k \in \mathbb{Z}_+$, $k \leq n$, $E \subset \mathbb{R}^n$ an open set where every closed k -form is exact, and $T : E \rightarrow U$ a C^2 invertible function with $U \subset \mathbb{R}^n$ open, every closed k -form in U is exact in U (i.e., every closed form is exact in any set which is C^2 -isomorphic to a convex open set).*

5 Lebesgue Theory

A family $\mathcal{R}$ of sets is a **ring** if $A, B \in \mathcal{R}$ implies

$$A \cup B \in \mathcal{R} \quad \text{and} \quad A \setminus B \in \mathcal{R},$$

and is a **σ -ring** if

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{R} \quad (\text{which implies } \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}).$$

Any function $\phi : \mathcal{R} \rightarrow \overline{\mathbb{R}}$ is a **set function**, and is **additive** if

$$A \cap B = \emptyset \implies \phi(A \cup B) = \phi(A) + \phi(B),$$

and is **countably additive** if

$$A_i \cap A_{j \neq i} = \emptyset \implies \phi \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \phi(A_n).$$

A set is **elementary** if it is the union of a finite number of intervals (n -cells). Let $\mathcal{E}$ denote the family of all elementary subsets of $\mathbb{R}^n$. Then a **regular** set function is a nonnegative additive set function ϕ of domain $\mathcal{E}$ such that

$$\forall A \in \mathcal{E}, \varepsilon > 0 \quad \exists \text{ closed } F, \text{ open } G \in \mathcal{E}, F \subset A \subset G, \quad \phi(G) - \varepsilon \leq \phi(A) \leq \phi(F) + \varepsilon.$$

A set X is a **measure space** if there exists a σ -ring $\mathfrak{M}$ of subsets X and a nonnegative countably additive set function μ defined over $\mathfrak{M}$. Every subset of X is a **measurable set**, and every element of $\mathfrak{M}$ is a **measurable space**. The sum of two measurable sets A and B is

$$A + B = \{x + y \mid x \in A, y \in B\}.$$

Note that

$$\sqrt[d]{m(A+B)} \geq \sqrt[d]{m(A)} + \sqrt[d]{m(B)},$$

always, and if a measurable set $E \subset \mathbb{R}^d$ is convex,

$$m((1+\lambda)E) = (1+\lambda)^d m(E).$$

5.1 Integration

An orthonormal set $\{\phi_n\}$ is **complete** if for $f \in L^2(\mu)$ the equations

$$\forall n, \quad \int_X f \overline{\phi_n} d\mu = 0$$

imply that $\|f\| = 0$.

Source: [SS05, p.120]

A measurable function f on $\mathbb{R}^d$ is **locally integrable** if

$$\forall r > 0, p \in \mathbb{R}^d, \quad \int |f(x) \chi_{B_r(p)}(x)| dx < \infty.$$

A point $x \in E \subset \mathbb{R}^d$ is a point of **Lebesgue density** of E if

$$0 < r < \epsilon, p \in \mathbb{R}^d, \quad \epsilon \rightarrow 0, x \in B_r(p) \implies \frac{m(B_r(p) \cap E)}{m(B_r(p))} = 1.$$

Similarly, the **Lebesgue set** $\mathcal{E}$ of $f \in L^1_{loc}(\mathbb{R}^d)$ is

$$\mathcal{E} = \{x \mid 0 < r < \epsilon, p \in \mathbb{R}^d, \quad \epsilon \rightarrow 0, x \in B_p(r) \implies \frac{1}{m(B_r(p))} \int_{B_r(p)} |f(y) - f(x)| dy = 0\}.$$

Hence points of continuity are members of the Lebesgue set.

A collection of sets $\{U_\alpha\}$ **shrinks regularly** (have **bounded eccentricity**) to x if there is exist some $r > 0, p \in \mathbb{R}^d, \beta > 0$ with $x \in B_r(p)$ such that

$$\forall \alpha, \quad U_\alpha \subset B_r(p) \quad \text{and} \quad \beta m(U_\alpha) \geq m(B_r(p)).$$

We identify the family of functions $\mathcal{K} = \{K_\delta \mid \delta > 0\}$ as **approximations to the identity**, where for all $x \in \mathbb{R}^d$,

$$1) \quad \int_{\mathbb{R}^d} K_\delta(x) dx = 1.$$

$$2) \quad K_\delta(x) \leq \frac{A}{\delta^d}.$$

$$3) \quad K_\delta(x) \leq \frac{A\delta}{x^{d+1}},$$

implying that $\delta \rightarrow 0$ yields $f \star K_\delta \rightarrow f$ in $L^p(\mathbb{R}^d)$. Then, loosely, as $\delta \rightarrow 0$, K_δ tends to $\mathcal{D}$, the **Dirac delta**:

$$\mathcal{D}(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}, \quad \int \mathcal{D}(x) dx = 1.$$

6 Trigonometric Functions

For

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z,$$

define

$$C(x) = \frac{1}{2} [E(ix) + E(-ix)], \quad S(x) = \frac{1}{2i} [E(ix) - E(-ix)],$$

yielding

$$|E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = 1, \quad C'(x) = -S(x), \quad S'(x) = C(x),$$

and for x_0 the smallest positive number such that $C(x_0) = 0$, define

$$\pi = 2x_0,$$

yielding

$$E\left(\frac{\pi i}{2}\right) = i, E(\pi i) = -1, \quad E(2\pi i) = 1, E(z + 2\pi) = E(z).$$

7 Differentiation

A curve γ is **rectifiable** if the sum of distances between adjacent points in a piecewise linear approximation is bounded. The **length** $L(\gamma)$ of such a curve is the infimum of the possible bounds. We then define the **variation** of a function $F : K \rightarrow \mathbb{C}^d$ for compact domain K for a particular partition to be the sum of distances between adjacent points. F is then of **bounded variation** if all such partitions have a supremum $M \in \mathbb{R}$, making R the **total variation** of f on K . We then have **positive variation** $P_F(a, x)$ and **negative variation** $N_F(a, x)$ for increasing and decreasing regions of the partition, respectively, i.e.,

$$R = P_F(a, x) - N_F(a, x)$$

for real-valued F .

Theorem 7.1 (3.2). *If a function F is of bounded variation on K , F is differentiable almost everywhere.*

A function F of domain $[a, b]$ is **absolutely continuous** if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^n (b_k - a_k) < \delta \implies \sum_{k=1}^N |F(b_k) - F(a_k)| < \epsilon$$

for disjoint intervals (a_k, b_k) , $k = 1, \dots, N$. We then have that absolutely continuous functions are differentiable almost everywhere and integrable, i.e.,

$$F(x) - F(a) = \int_a^x F'(y) dy, \quad a \leq x \leq b.$$

Conversely, if f is integrable on $[a, b]$, there exists an absolutely continuous function F such that $F'(x) = f(x)$ almost everywhere, where we may take $F(x) = \int_a^x f(y) dy$.

Theorem 7.2. *Bounded increasing functions have at most countably many discontinuities.*

A function f is **Lipchitz continuous** if

$$|f(x) - f(y)| \leq K|y - x|.$$

7.1 Minkowski

A parametrized curve $z(t) = (x(t), y(t))$ is **simple** if $t \mapsto z(t)$ is injective. $z(t)$ is a **closed simple curve** if it is injective excepting $z(a) = z(b)$. Finally, $z(t)$ is **quasi-simple** if the mapping is injective for t in the complement of finitely many points in $[a, b]$. Let Γ denote the curve (the image of the parametrization), and Γ^δ the set of points within an open ball of radius δ around some point of Γ . The **Minkowski content** of Γ is then

$$\mathcal{M}(\Gamma) = \lim_{\delta \rightarrow 0} \frac{m(\Gamma^\delta)}{2\delta}.$$

Theorem 7.3 (4.4). *Every quasi-simple curve Γ has Minkowski content $\mathcal{M}$ iff Γ is rectifiable, and $\mathcal{M}$ is the length of Γ .*

Theorem 7.4 (4.8, Isoperimetric inequality). *For $\Omega \subset \mathbb{R}^2$ open and bounded with boundary $\Gamma = \partial\Omega$,*

$$4\pi m(\Omega) \leq \limsup_{\delta \rightarrow 0} \frac{m(\Gamma^\delta)^2}{2\delta}.$$

8 Hilbert

$$\ell^2(\mathbb{Z}) = \left\{ (x_i) \mid \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty \right\}$$

Every separable infinite-dimensional Hilbert space is isomorphic to $\ell^2(\mathbb{Z})$.

The **radial limit** for a function F over the unit disc $\mathbb{D}$ at point $\theta \in [-\pi, \pi]$ is given by

$$\lim_{r \rightarrow 1^-} F(re^{i\theta}).$$

Theorem 8.1 (3.1, Fatou's Theorem). *For $F(re^{i\theta})$ bounded and holomorphic on the unit disc, F has radial limits a.e. θ .*

The **Hardy space** $H^2(\mathbb{D})$ is every holomorphic function F on the unit disc $\mathbb{D}$ with

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^2 d\theta < \infty.$$

The **Hermite functions** $h_k(x)$ are defined by the generating identity

$$\sum_{k=0}^{\infty} h_k(x) \frac{t^k}{k!} = e^{-\left(\frac{x^2}{2} - 2tx + t^2\right)}.$$

The n^{th} **Legendre polynomial** φ_n is given by

$$\varphi_n(x) = \left(\frac{d}{dx} \right)^n (1 - x^2)^n.$$

A continuous function F defined on $\mathbb{R}$ is **almost periodic** if it is the uniform limit of elements in $\mathcal{H}_0$. A sequence $\{f_{n_k}\}$ **converges weakly** to f if $\|f_{n_k}\| = 1$ for all n , and for all $g \in \mathcal{H}$,

$$\lim_{k \rightarrow \infty} \langle f_{n_k}, g \rangle = \langle f, g \rangle.$$

$F = F_A + F_C + F_J$ is the **Lebesgue decomposition** of F if each function is increasing, F_A is absolutely continuous, F_C is continuous with $F'_C(x) = 0$ a.e x , and F_J is a jump function.

8.1 Operators

If there exists an operator $T \in \mathcal{L}(L^2(\mathbb{R}^d))$ with

$$T(f)(x) = \int_{\mathbb{R}^d} K(x, y) f(y) dy,$$

then such an operator T is an **integral operator** with **kernel** K . **Hilbert-Schmidt operators** are operators with a kernel $K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. By Fubini's theorem, $y \mapsto K(x, y)f(y)$ is integrable a.e. x . We also find easily that $\|T\| \leq \|K\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$, and that T^* has kernel $\overline{K(y, x)}$. $T \in \mathcal{L}(\mathcal{H})$ is **compact** if $X \subset \mathcal{H}$ implies that $T(X)$ is a sequentially compact set.

Theorem 8.2 (6.2, Spectral). *For $T \in \mathcal{L}(\mathcal{H})$ compact, and self-adjoint (i.e., symmetric/Hermitian), there exists an orthonormal eigenbasis $\{\varphi_k\}_{k=1}^\infty$ of T , with all real eigenvalues tending in norm to 0 as $k \rightarrow \infty$.*

The collection of eigenvalues above $(\{\lambda_k\})$ is the **spectrum** of T .

8.2 Fourier

The **Fourier transform** of a function f over $\mathbb{R}^d$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

with inversion

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

The **Schwartz class** $\mathcal{S}(\mathbb{R}^d)$ is every smooth function f with

$$x^k \left(\frac{\partial}{\partial x} \right)^i f, \quad k, i \in \mathbb{Z}_+, i \leq d$$

always bounded on $\mathbb{R}^d$.

8.3 Hardy

Let $\mathbb{C}^+ = \{z \mid z = a + bi, a \in \mathbb{R}, b \in \mathbb{R}^+\}$ denote the **upper half-plane**. The **Hardy space** $H^2(\mathbb{C}^+)$ is defined to be every holomorphic function F over $\mathbb{C}^+$ with

$$\|F\|_{H^2(\mathbb{C}^+)}^2 = \sup_{b \in \mathbb{R}_+} \int_{\mathbb{R}} |F(a + bi)|^2 dx < \infty.$$

Theorem 8.3. *For $f \in H^2(\mathbb{C}^+)$,*

$$\lim_{b \rightarrow 0} F(x + bi) = F_0(x)$$

as a limit in the $L^2(\mathbb{R})$ norm, and as a limit for a.e. x .

Such a function F_0 is the **boundary-value function** of f . In proving this we may use the **Poisson integral representation**

$$\int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi|\xi|y} e^{2\pi i x \xi} d\xi = \int_{\mathbb{R}} f(x - t) \mathcal{P}_y(t) dt, \quad \text{where } \mathcal{P}_y(x) = \frac{1}{\pi} \cdot \frac{y}{y^2 + x^2},$$

the **Poisson kernel**.

For P the orthogonal projection onto the subspace $\mathcal{S}$ of L^2 , and F the unique element in $H^2(\mathbb{C}^+)$ with boundary function $P(f)$, then we may compute the **Cauchy integral**

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z} dt, \quad z \in \mathbb{C}^+.$$

A **Fourier multiplier operator** T on $\mathbb{R}$ is a linear operator on $L^2(\mathbb{R})$ determined by a bounded function m , the **multiplier**, such that

$$\widehat{(Tf)}(\xi) = m(\xi)\hat{f}(\xi).$$

The orthogonal projection P as defined above would have multiplier characteristic function $\chi(\xi)$.

The **Hilbert transform** H is given by

$$P = \frac{I + iH}{2},$$

thence being a Fourier multiplier operator with multiplier

$$\frac{1}{i}\text{sign}(\xi).$$

8.4 Linear PDEs

Define linear partial differential equation

$$L(u) = f,$$

where the operator L has

$$4L = \sum_{|\alpha| \leq n} a_{\alpha} \left(\frac{\partial}{\partial x} \right)^{\alpha}, \quad a_{\alpha} \in \mathbb{C}.$$

Suppose one seeks to solve this equation with premise that $u, f \in \mathcal{S}$. Then we reformulate the problem into the algebraic equation

$$P(\xi)\hat{u}(\xi) = \hat{f}(\xi),$$

where $P(\xi)$ is the **characteristic polynomial** of f defined by

$$P(\xi) = \sum_{|\alpha| \leq n} a_{\alpha} (2\pi i \xi)^{\alpha}$$

since by the Fourier transform identity a solution $u \in \mathcal{S}$ must be uniquely determined by

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{P(\xi)}.$$

Theorem 8.4 (3.2). *For $\Omega \subset \mathbb{R}^d$ open and bounded and a linear partial differential operator L of constant coefficients, there exists a bounded linear operator K on $L^2(\Omega)$ with*

$$L(Kf) = f$$

in the weak sense (i.e., $u = K(f)$ is a weak solution to $L(u) = f$).

We say a differential operator L is **elliptic** if

$$|P(\xi)| \geq c|\xi|^n \quad \text{for some } c > 0 \text{ and all } \xi \text{ sufficiently large.}$$

The **Mellin transform** is given by

$$\mathcal{M}f(x) = \int_0^\infty f(t)t^{ix-1} dt$$

for functions f of compact support in $\mathbb{R}_+$.

8.5 Dirichlet Principle

The **Dirichlet problem** is the search for a function $u(x_1, x_2)$ for a bounded and open set $\Omega \subset \mathbb{R}^2$ and a continuous function $f : \partial\Omega \rightarrow \mathbb{C}$ with $\Delta u = 0$ in Ω , and $u = f$ on $\partial\Omega$.

The **Dirichlet integral** is given by

$$\mathcal{D}(U) = \int_\Omega |\nabla U|^2 = \int_\Omega \left| \frac{\partial U}{\partial x_1} \right|^2 + \left| \frac{\partial U}{\partial x_2} \right|^2 dx_1 dx_2.$$

A function u is **weakly harmonic** in Ω if

$$\langle u, \Delta\psi \rangle = 0, \quad \psi \in C_c^\infty(\Omega).$$

A continuous function u over Ω has the **mean-value property** if

$$u(x_0) = \frac{1}{m(B)} \int_B u(x) dx$$

for every ball B with center x_0 with closure $\overline{B} \subset \Omega$.

Theorem 8.5 (4.2). *If u is harmonic in Ω , then u satisfies the mean-value property. Also, any continuous function satisfying the mean-value property is harmonic.*

Theorem 8.6 (4.12). *For $\Omega \subset \mathbb{R}^2$ open, bounded, and satisfying the outside-triangle condition (existence of an isosceles triangle whose vertex may be used as outward normal), if f is continuous on $\partial\Omega$, then the boundary value problem $\Delta u = 0$ with u continuous in $\overline{\Omega}$ and $u|_{\partial\Omega} = f$ is always uniquely solvable.*

9 Measure and Integration

A set $E \subset X$ is **Carathéodory measurable** (**measurable**) if

$$\mu_*(A) = \mu_*(E \cap A^C) + \mu_*(E^C \cap A)$$

for every $A \subset X$.

An exterior measure μ_* on metric space (X, d) is a **metric exterior measure** if

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$$

for $d(A, B) > 0$.

The **Borel σ -algebra** $\mathcal{B}_X$ is the intersection of every σ -algebra containing the open sets of X . Elements of $\mathcal{B}_X$ are **Borel sets**. Any measure defined on $\mathcal{B}_X$ is a **Borel measure**.

An **algebra** in X is a non-empty collection of subsets of X that is closed under complements, finite unions, and finite intersections.

For an algebra $\mathcal{A}$ in X , a **premeasure** $\mu_0 : \mathcal{A} \rightarrow [0, \infty]$ satisfies nullity and disjoint countability.

The **product measure** between is defined between measure spaces $(X_1, \mathcal{M}_1\mu_1)$ and $(X_2, \mathcal{M}_2\mu_2)$ $\mu_1 \times \mu_2$ on $X_1 \times X_2$. We then have an abstract Fubini:

Theorem 9.1 (3.3). *For $f(x_1, x_2)$ is an integrable function on $(X_1 \times X_2, \mu_1 \times \mu_2)$, we have that for a.e. $x_2 \in X_2$, the slice $f^{x_2}(x_1) = f(x_1, x_2)$ is integrable on (X_1, μ_1) , $\int_{X_1} f(x_1, x_2) d\mu_1$ is an integrable function on X_2 , and $\int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f d\mu_1 \times \mu_2$.*

We may define the **Lebesgue-Stieltjes integral** for F an increasing **normalized** (right-continuous) function on a closed interval $[a, b]$ extended to $\mathbb{R}$ by setting $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$ with resulting measure μ having intervals $(-\infty, a]$ and (b, ∞) with measure zero, and

$$\int_{\mathbb{R}} f(x) d\mu(x) = \int_a^b f(x) dF(x)$$

for every f that is integrable with respect to μ .

A **signed measure** ν on a σ -algebra $\mathcal{M}$ is satisfies $-\infty < \nu(E) \leq \infty$ for all $E \in \mathcal{M}$ and disjoint countable additivity. We then label the function $|\nu|$ on $\mathcal{M}$ the **total variation** of ν by

$$|\nu|(E) = \sup \sum_{j=1}^{\infty} |\nu(E_j)|,$$

where the supremum is taken over all partitions of E . Then ν is **σ -finite** if the measure $|\nu|$ is σ -finite. A measure ν is **supported** on a set A if $\nu(E) = \nu(E \cap A)$ for all $E \in \mathcal{M}$.

Signed measures ν and μ on $(X, \mathcal{M})$ are **mutually singular** if there are disjoint subsets A and B in $\mathcal{M}$ so that

$$\nu(E) = \nu(A \cap E) \text{ and } \mu(E) = \mu(B \cap E) \text{ for all } E \in \mathcal{M}$$

and denote such with $\nu \perp \mu$. We then say ν is **absolutely continuous** with respect to μ if

$$\nu(E) = 0$$

whenever $E \in \mathcal{M}$ and $\mu(E) = 0$ and denote such with $\nu \ll \mu$. Note that if $f \in L^1(X, \mu)$, then the signed measure ν defined by

$$\nu(E) = \int_E f d\mu$$

is absolutely continuous with respect to μ .

Theorem 9.2 (4.3). *For μ a σ -finite positive measure on the measure space $(X, \mathcal{M})$ and ν a σ -finite signed measure on $\mathcal{M}$, there exist unique signed measures ν_a and ν_s on $\mathcal{M}$ such that $\nu_a \ll \mu$, $\nu_s \perp \mu$ and $\nu = \nu_a + \nu_s$. Further, the measure ν_a takes the form $d\nu_a = f d\mu$; i.e.,*

$$\nu_a(E) = \int_E f(x) d\mu(x)$$

for some extended μ -integrable function f .

Theorem 9.3 (5.2, Mean ergodic theorem). *For T an isometry of the Hilbert space $\mathcal{H}$, and P be the orthogonal projection on the subspace of the invariant vectors of T , let $A_n = \frac{1}{n}(\sum_{i=0}^{n-1} T^i)$. Then for each $F \in \mathcal{H}$, $A_n(f)$ converges to $P(f)$ in norm as $n \rightarrow \infty$.*

Theorem 9.4 (5.3). *For $f \in L^1(X, \mu)$, the maximal function*

$$f^*(x) = \sup_{1 \leq m < \infty} \frac{1}{m} \sum_{k=0}^{m-1} |f(\tau^k(x))|$$

is finite a.e. x . Further, there exists $A \in \mathbb{R}$ such that

$$\mu(\{x \mid f^*(x) > \alpha\}) \leq \frac{A}{\alpha} \|f\|_{L^1(x, \mu)}$$

for all $\alpha > 0$.

Theorem 9.5 (5.4, Pointwise (individual) ergodic theorem). *For f integrable over X , the averages*

$$A_m(f) = \frac{1}{m} \sum_{k=0}^{m-1} f(\tau^k(x))$$

converge to a limit as $m \rightarrow \infty$ a.e. $x \in X$.

For a measure-preserving transformation τ on the space (X, μ) , such a transformation is **mixing** if whenever E and F are measurable subsets, then

$$\mu(\tau^{-n}(E) \cap F) = \mu(E)\mu(F)$$

as $n \rightarrow \infty$.

The **spectral resolution** (or **spectral family**) on a Hilbert space $\mathcal{H}$ is a function $\lambda \mapsto E(\lambda)$ from $\mathbb{R}$ to orthogonal projections on $\mathcal{H}$ which satisfies that $\|E(\lambda)f\|$ is always an increasing function of λ , there is an interval $[a, b]$ such that $\lambda < a \implies E(\lambda) = 0$, and $\lambda \geq b \implies E(\lambda) = I$, and that $E(\lambda)$ is right-continuous with $E(\mu^+)f = E(\lambda)f$.

Theorem 9.6 (6.1). *For T a bounded symmetric operator on $\mathcal{H}$, there exists a spectral resolution $\{E(\lambda)\}$ with*

$$T = \int_{a^-}^b \lambda dE(\lambda)$$

so

$$\langle Tf, g \rangle = \int_{a^-}^b \lambda d\langle E(\lambda)f, g \rangle = \int_{a^-}^b \lambda dF(\lambda).$$

10 Hausdorff, Fractals

For $E \subset \mathbb{R}^d$, the **exterior α -dimensional Hausdorff measure of E** is given by

$$m_\alpha^*(E) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_k (\text{diam } F_k)^\alpha \mid E \subset \bigcup_{k=1}^{\infty} F_k, \quad \text{diam } F_k \leq \delta \text{ for all } k \right\}.$$

E then has **Hausdorff dimension**

$$\alpha = \sup\{\beta \mid m_\beta(E) = \infty\} = \inf\{\beta \mid m_\beta(E) = 0\}$$

and write $\alpha = \dim E$, and say E has **strict Hausdorff dimension** if $0 < m_\alpha(E) < \infty$.

A function f satisfies a **Lipschitz condition with exponent γ** (i.e., is **Hölder γ**) if

$$|f(x) - f(y)| \leq M|x - y|^\gamma.$$

The **Minkowski dimension** of E is

$$\inf\{\beta \mid m(E^\delta) = O(\delta^{d-\beta}) \text{ as } \delta \rightarrow 0\}.$$

Theorem 10.1 (2.9). *For $S_1, S_2, \dots, S_m$ m similarities, with the same ratio r with $0 < r < 1$, there exists a unique non-empty compact set F such that*

$$F = S_1(F) \cup \dots \cup S_m(F).$$

The **Radon transform** of f is given by

$$\mathcal{R}(f)(t, \gamma) = \int \mathcal{P}_{t, \gamma} f.$$

Theorem 10.2 (4.5). *For f absolutely continuous in $\mathbb{R}^{d \geq 3}$,*

$$\int_{S^{d-1}} \mathcal{R}^*(f)(\gamma) d\sigma(\gamma) \leq c \left[\|f\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)} \right]$$

for a constant $c > 0$ independent of f .

Part II

Complex

[SS03a] A function f is **holomorphic (regular)** when it is differentiable in the complex sense:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}, \quad (h \in \mathbb{C}).$$

It then follows that for any closed path γ in $\Omega \subset \mathbb{C}$,

$$\int_{\gamma} f(z) dz = 0,$$

and for any two holomorphic functions f and g over Ω which are equal in an arbitrarily small ball in Ω , then $f = g$ everywhere in Ω .

The **zeta function** is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and is holomorphic for $\operatorname{Re}(s) > 1$.

The **theta function** is given by

$$\Theta(z \mid \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z},$$

and is holomorphic for all z but only for τ for $\operatorname{Im}(\tau) > 0$.

Part III

Fourier

[SS03b]

Our investigation is led by the **wave equation**, which gives us partial differential equations using cosine and sine, and the **heat equation**, which gives us the Laplacian.

For a sequence $\{x_n\}$ define the k^{th} **Cesáro mean** of the sequence as

$$\sigma_k = \frac{1}{k} \sum_{n=0}^{k-1} x_n,$$

and the sequence $\{x_n\}$ is **Cesáro summable** if the limit of the mean as $k \rightarrow \infty$ exists.

The k^{th} **Fejér kernel** by the k^{th} Cesáro mean of the sequence of Dirichlet kernels. For $\{x_n\}$ a sequence of complex numbers, we say such a sequence is **Abel summable** to s if for $0 \leq r < 1$,

$$A(r) = \sum_{k=0}^{\infty} x_k r^k$$

exists and

$$\lim_{r \rightarrow 1} A(r) = s.$$

Every $A(r)$ is an **Abel mean** of the series.

The **Bernoulli numbers** B_n are given by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

and the **Bernoulli polynomials** $B_n(x)$ by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}.$$

A sequence of numbers $\{\xi_i\}$ is **equidistributed** if for every $(a, b) \subset [0, 1)$,

$$\lim_{N \rightarrow \infty} \frac{|\{1 \leq n \leq N \mid \xi_n \in (a, b)\}|}{N} = b - a.$$

Unsurprisingly, for any irrational γ , the set $\mathbb{Z} \cdot \gamma \pmod{1}$ is equidistributed in $[0, 1)$.

A Fourier series which skips many terms is a **lacunary Fourier series**.

11 Transform

Note that for $f(x) = e^{-\pi x^2}$, the Gaussian,

$$\hat{f}(\xi) = f(\xi).$$

Theorem 11.1 (1.12, Plancherel). *For $f \in \mathcal{S}(\mathbb{R})$,*

$$\|\hat{f}\| = \|f\|.$$

Theorem 11.2 (2.1). *For the **heat equation** and **heat kernel***

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \mathcal{H}_t(x) = K_\delta(x), \quad (\delta = 4\pi t),$$

$f \in \mathcal{S}(\mathbb{R})$, and

$$u(x, t) = (f \star \mathcal{H}_t)(x), \quad t \in \mathbb{R}_+,$$

the function u is $C^2(\mathbb{R}, \mathbb{R}_+)$ and u solves the heat equation, and $u(x, t) \rightarrow f(x)$ uniformly as $t \rightarrow 0$.

Theorem 11.3 (2.6). *For $f \in \mathcal{S}(\mathbb{R})$, $u(x, y) = (f \star \mathcal{P}_y)(x)$, $u(x, y) \in C^2(\mathbb{R}_+^2)$ and $\Delta u = 0$, and $u(x, y) \rightarrow f(x)$ uniformly as $y \rightarrow 0$.*

Define the **periodization** F of a function $f \in \mathcal{S}(\mathbb{R})$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + n).$$

Theorem 11.4 (3.1, Poisson Summation). *For $f \in \mathcal{S}(\mathbb{R})$,*

$$\sum_{n=-\infty}^{\infty} f(x + n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

The **Cauchy problem** for the wave equation is given by

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x),$$

for $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Theorem 11.5 (3.1). *A solution to the Cauchy problem is*

$$u(x, t) = \int_{\mathbb{R}^d} \left[\hat{f}(\xi) \cos(2\pi|\xi|t) + \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} \right] e^{2\pi i x \cdot \xi} d\xi.$$

The **Bessel function of order** $n \in \mathbb{Z}$ is given by

$$J_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\rho \sin \theta} e^{-in\theta} d\theta.$$

For the Radon transform R ,

$$\hat{\mathcal{R}}(f)(s, \gamma) = \hat{f}(s\gamma).$$

The **dual Radon transform** is given by

$$\mathcal{R}^*(F)(x) = \int_{S^2} F(x \cdot \gamma, \gamma) d\sigma(\gamma).$$

Further, for any $f \in \mathcal{S}(\mathbb{R}^3)$,

$$\Delta(\mathcal{R}^*\mathcal{R}(f)) = -8\pi^2 f.$$

12 Finite

Consider the group

$$\mathbb{Z}(N) \cong N\mathbb{Z}/N$$

and the corresponding orthogonal basis for the vector space of Riemann-integrable functions on $\mathbb{Z}(N)$

$$\{e_\ell\} = \{e^{2\pi ki\theta/N}\}_{k=0}^{N-1}$$

with inner product

$$\langle F, G \rangle = \sum_{k=0}^{N-1} F(k) \overline{G(k)}$$

and consequent n^{th} Fourier coefficient

$$a_n = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{-2\pi i kn/N}.$$

A **character** $e : G \rightarrow S^1(\mathbb{C})$ on a finite abelian group G is a function such that

$$e(a \cdot b) = e(a)e(b),$$

i.e., a homomorphism to the circle group. The **trivial (unit) character** has $e(G) = 1$. $\hat{G}$ is the set of all characteristics for G , which is also an abelian group, is called the **dual group** of G .

Theorem 12.1 (2.3). *For V the vector space of complex-valued functions over a finite abelian group G with Hermitian inner product*

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{a \in G} f(a) \overline{g(a)},$$

the characters of G form an orthonormal family with respect to such an inner product.

13 Dirichlet

Theorem 13.1 (1.10). *For every $s > 1$,*

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$

where the product is taken over all primes.

Theorem 13.2 (2.1, Dirichlet's Theorem). *For q and l coprime, there are infinitely many primes of the form $l \equiv 0 \pmod{q}$.*

Extensions of characters $e \in \hat{G}$ to all of $\mathbb{Z}$ given by

$$\chi(m) = \begin{cases} e(m) & \text{if } m \text{ and } q \text{ are coprime} \\ 0 & \text{otherwise} \end{cases}$$

are **Dirichlet characters mod q** . **L -functions** are then defined for $s > 1$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character.

Theorem 13.3 (2.4). *For $s > 1$,*

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{(1 - \chi(p)p^{-s})},$$

where the product is over all primes.

For these theorems we make use of the remaining definitions and statements:

$$\log_a \left(\frac{1}{1-z} \right) = \sum_{k=1}^{\infty} \frac{z^k}{k} \quad \text{for } |z| < 1, \quad \log_b(L(s, \chi)) = - \int_s^{\infty} \frac{L'(t, \chi)}{L(t, \chi)} dt.$$

Theorem 13.4 (3.7). *If $\chi \neq \chi_0$, then $L(1, \chi) \neq 0$.*

The hyperbolic sum of

$$S_N = \sum \sum F(m, n)$$

for $m, n \in \mathbb{Z}_+$, $mn \leq N$,

$$S_N = \sum_{1 \leq k \leq N} \left(\sum_{nm=k} F(m, n) \right).$$

Part IV

Functional

[SS11]

14 L^p

The **Sobolev spaces** $L_k^p(\mathbb{R}^d)$ may be defined as $L^p(\mathbb{R}^d)$ functions with weak derivatives up to degree k .

If $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then p and q are **complex (dual) exponents**.

Theorem 14.1 (1.1, Hölder). *For conjugate exponents p and q , $f \in L^p$, $g \in L^q$, and $fg \in L^1$,*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

For $X = \mathbb{Z}$ equipped with the counting measure, the converse holds:

$$L^p(\mathbb{Z}) \subset L^q(\mathbb{Z}) \quad \text{and} \quad \|f\|_{L^q} \leq \|f\|_{L^p} \quad \text{if } p \leq q.$$

Theorem 14.2 (1.2, Minkowski). *For $p \in [1, \infty)$ and $f, g \in L^p$, we have*

$$f + g \in L^p \quad \text{and} \quad \|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Theorem 14.3 (2.1). *The vector space L^∞ equipped with norm $\|\cdot\|_{L^\infty}$ (the **essential-supremum**) is a complete vector space.*

Theorem 14.4. *For finite conjugate exponents p and q ,*

$$(L^p)^\star \cong L^q \quad \text{and} \quad \|v'\|_{(L^p)^\star} = \|g\|_{L^q},$$

where $v' \in (L^p)^\star$ bounded and $g \in L^q$ uniquely satisfies $v'(f) = \int_X f(x)g(x) d\mu(x)$ for all $f \in L^p$, justifying the term “dual” for conjugate exponents.

The isomorphism from a vector space to its dual fails for L^∞ because injectivity is impossible from $(L^\infty)^\star$.

A **proper hyperplane** is a linear subspace of V which arises as the kernel of a non-zero linear functional on V (i.e., a subspace which spans V with the addition of any vector not in V). We may define a hyperplane H of parameters $v' \in V^\star$ and $a \in \mathbb{F}$ with

$$H = \{v \in V \mid v'(v) = a\}.$$

Theorem 14.5 (5.2, Hahn-Banach). *For a subspace U of a vector space V and a function $p : V \rightarrow \mathbb{F}$ and $v'_0 \in V^\star$ with*

$$v'_0(v) \leq p(v), \quad \forall v \in U,$$

v'_0 can be extended to a $v' \in V^\star$ with

$$v'(v) \leq p(v), \quad \forall v \in V.$$

The **dual transformation** of an operator $T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is given by

$$(T^\star v'_2)(v_1) = v'_2(Tv_1), \quad v'_2 \in \mathcal{B}_2^\star, v'_1 \in \mathcal{B}_1^\star, v_1 \in \mathcal{B}_1.$$

When $\mathbb{F} = \mathbb{C}$, the counting measure-converse of the Hölder inequality uses f with definition

$$f(x) = |g(x)|^{q-1} \frac{\overline{\text{sign } g(x)}}{\|g\|_{L^q}^{q-1}},$$

where $\text{sign } z = \frac{z}{|z|}$.

Theorem 14.6 (7.4). *For a metric space X , positive $v' \in C^\star(X)$ (WLOG $v'(1) = 1$), and $\varepsilon > 0$, there exists compact set $K_\varepsilon \subset X$ with*

$$|v'(v)| \leq \sup_{x \in K_\varepsilon} |v(x)| + \varepsilon \|v\| \quad \forall v \in C(X)$$

(unnecessary if X is complete and separable), and there exists a unique positive Borel measure μ with

$$v'(v) = \int_X v(x) d\mu(x) \quad \forall v \in C(X),$$

where C denotes the vector space of continuous, bounded functions with the supremum norm, a result analogous to Riesz representation.

We say an L^p space has **mixed norm** when we have an underlying product space such as $\{(x, t)\} = \mathbb{R}^d \times \mathbb{R}$ with product measure $dx dt$ and take $L_t^r(L_x^p) = L^{p,r}$ to be every function f where

$$\|f\|_{L^{p,r}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |f(x, t)|^p dx \right)^{\frac{r}{p}} dt \right)^{\frac{1}{r}} < \infty.$$

Young's inequality denotes

$$\|f \star g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r} \quad \text{whenever} \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1,$$

or where for continuous and strictly increasing functions φ and ψ defined over nonnegative reals,

$$\Phi(x) = \int_0^x \varphi(u) du, \quad \text{and} \quad \Psi(x) = \int_0^x \psi(u) du,$$

we have that

$$ab \leq \Phi(a) + \Psi(b)$$

for all nonnegative a and b .

The **Orlicz spaces** are defined by

$$L^\Phi = \left\{ f \mid f \text{ is measurable and } \int_X \Phi \left(\frac{|f(x)|}{M} \right) d\mu < \infty \text{ for some } M \in \mathbb{R}_+ \right\}$$

and

$$\|f\|_\Phi = \inf_{M \in \mathbb{R}_+} \int_X \Phi \left(\frac{|f(x)|}{M} \right) d\mu \leq 1$$

for $\Phi(t)$ continuous, convex, increasing, defined on nonnegative reals, and self-affine ($\Phi(0) = 0$).

15 Harmonic

The **conjugate function** or **allied series** for the Fourier series of an L^2 function f represents the relation

$$\tilde{f}(\theta) \sim \sum_{n=-\infty}^{\infty} \frac{\text{sign}(n)}{i} e^{in\theta},$$

having the significance of

$$\frac{1}{2}(f(\theta) + i\tilde{f}(\theta) + a_0) \sim \sum_{n=0}^{\infty} a_n e^{in\theta} = F(e^{i\theta}),$$

where $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is the analytic function in the open unit disc given by the Cauchy projection of f ,

$$F(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\theta)}{e^{i\theta} - z} i e^{i\theta} d\theta.$$

Theorem 15.1 (2.1, Riesz Interpolation). *For an operator $T \in \mathcal{L}(V_{\alpha,\beta}, V_{\rho,\tau})$ with*

$$L_{p,q} = \{f_p + f_q \mid f_p \in L^p, f_q \in L^q\}, \quad \sup \frac{\|f\|_{L^\alpha}}{\|T(f)\|_{L^\beta}} < \infty, \quad \sup \frac{\|f\|_{L^\rho}}{\|T(f)\|_{L^\tau}} < \infty,$$

(i.e., T is bounded from L^β to L^α with norm γ and from L^τ to L^ρ with norm δ), then T is bounded from L^p to L^q with norm $\kappa \leq \gamma^{1-t} \delta^t$ whenever

$$\frac{1}{p} = \frac{1-t}{\alpha} + \frac{t}{\rho}, \quad \frac{1}{q} = \frac{1-t}{\beta} + \frac{t}{\tau}, \quad \text{and} \quad 0 \leq t \leq 1.$$

The **Riesz diagram** of an operator $T \in \mathcal{L}(V)$ is the set of pairs (p, q) in the real unit square satisfying

$$\sup \frac{\|f\|_{L^p}}{\|Tf\|_{L^q}} < \infty$$

for all simple f , making T of **type** (p, q) . Every Riesz diagram is convex, and $\log \|T\|_{p \rightarrow q}$ is a convex function on such a set.

For $f \in L^2(\mathbb{R})$, the Cauchy projection is holomorphic in the upper half-plane, and we are able to define the Hilbert transform by

$$(Hf)(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{\text{sign}(\xi)}{i} e^{2\pi i x \xi} d\xi,$$

implying that $H^2 = -I$, and that

$$(Hf)(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{|t| \geq \epsilon} f(x-t) \frac{dt}{t}.$$

Theorem 15.2 (3.2, Riesz). *For $p \in (1, \infty)$, the Hilbert transform H over $L^2 \cap L^p$ has*

$$\sup_{f \in L^2 \cap L^p} \frac{\|Hf\|_{L^p}}{\|f\|_{L^p}} < \infty$$

for all $f \in L^2 \cap L^p$ with a unique extension to all of L^p satisfying the same bound.

Theorem 15.3 (4.1). For $f \in L^p(\mathbb{R}^d)$, $p \in (1, \infty]$,

$$\sup_{f \in L^p} \frac{\|f^\star\|_{L^p}}{\|f\|_{L^p}} < \infty.$$

For any nonnegative, measurable function F , its **distribution function**, $\lambda(\alpha) = \lambda_F(\alpha)$, is defined by

$$\lambda(\alpha) = m(\{x \mid F(x) > \alpha\})$$

for all $\alpha > 0$.

A bounded, measurable function $\mathcal{F}$ over $\mathbb{R}^d$ is a p -**atom** associated to a ball $B \subset \mathbb{R}^d$ if $\mathcal{F}$ is supported in B , $|\mathcal{F}(x)|m(B) \leq (m(B))^{\frac{1}{p}}$ for all x , and $\int_{\mathbb{R}^d} \mathcal{F}(x) dx = 0$.

Define the **Hardy space for $p = 1$** by

$$\mathbf{H}_r^1(\mathbb{R}^d) = \left\{ f \mid f \in L^1, f = \sum_{k=1}^{\infty} \lambda_k k, k \text{ is an atom}, \lambda_k \in \mathbb{F}, \sum_{k=1}^{\infty} |\lambda_k| < \infty \right\}.$$

Theorem 15.4 (5.4). For $f \in \mathbf{H}_r^1(\mathbb{R}^d)$,

$$H_\epsilon(f) \in L^1(\mathbb{R}), \quad \text{and} \quad \sup \frac{\|H_\epsilon(f)\|_{L^1(\mathbb{R})}}{\|f\|_{\mathbf{H}_r^1(\mathbb{R}^d)}} < \infty \quad \text{as } \epsilon \rightarrow \infty.$$

Define the **maximal function** M for a bounded function Φ of compact support and $\Phi_\epsilon = \epsilon^{-d}\Phi(\frac{x}{\epsilon})$ and an a function $f \in L^1$ by

$$M(f)(x) = \sup_{\epsilon > 0} |(f \star \Phi_\epsilon)(x)|.$$

Theorem 15.5 (6.1). For $\Phi \in C_c^1(\mathbb{R}^d)$ and $M(f) \in L^1(\mathbb{R}^d)$,

$$f \in \mathbf{H}_r^1(\mathbb{R}^d) \implies \sup \frac{\|M(f)\|_{L^1(\mathbb{R})}}{\|f\|_{\mathbf{H}_r^1(\mathbb{R}^d)}} < \infty.$$

We say $f \in L_{\text{loc}}^1(\mathbb{R}^d)$ is of **bounded mean oscillation (BMO)** if

$$\sup \frac{1}{m(B)} \int_B |f(x) - f_B| dx < \infty,$$

for all balls B , where $f_B = \frac{1}{m(B)} \int_B f(x) dx$.

Theorem 15.6 (6.2). For $g \in \text{BMO}$, the $v' \in (\text{BMO})^\star$ defined by

$$v'(f) = \int_{\mathbb{R}^d} f(x)g(x) dx, \quad f \in \mathbf{H}_0^1,$$

has a unique extension to $\mathbf{H}_r^1$ that satisfies

$$\|v'\| \leq c\|g\|_{\text{BMO}},$$

and every bounded linear functional $v' \in (\mathbf{H}_r^1)^\star$ can be written the same way with

$$\|g\|_{\text{BMO}} \leq c'\|v'\|.$$

16 Distributions

The **test functions** $\mathcal{D}(\Omega)$ of the larger class of distributions are functions which belong to $C_c^\infty(\Omega)$, and a sequence $\{\varphi_n\} \subset \mathcal{D}$ **converges** to a $\varphi \in \mathcal{D}$ if the supports of the φ_n are contained in a common compact set, and for each multi-index α we have

$$\partial_x^\alpha \varphi_n \rightarrow \partial_x^\alpha \varphi$$

uniformly in x as $n \rightarrow \infty$. Thence, a **distribution** $F : \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ is a complex-valued linear functional which respects convergence. The **support of a distribution** is defined as the (closed) set of functions where either

$$F(\varphi) \neq 0 \quad \text{or} \quad \exists \varphi \in M \subset \mathcal{D}, \quad \text{supp } \varphi \not\subset M.$$

We denote the vector space of distributions on Ω by $\mathcal{D}'(\Omega)$, the dual of the vector space of complex-valued, indefinitely differentiable functions of compact support in $\Omega \subset \mathbb{R}^d$ open. We get natural expressions of the Dirac delta function and derivatives of differentiable functions in terms of distributions.

Theorem 16.1 (1.3). *For a distribution F of support C_1 , $\psi \in \mathcal{D}$ of support C_2 , the support of $F \star \psi$ is contained in $C_1 + C_2$.*

In the above we may define $F \star \psi$ as a function of x given by $F(\tilde{\psi}_x)(y) = \psi(x - y)$ or the distribution determined via $(F \star \psi)(\varphi) = F(\tilde{\psi}_x \star \varphi)$, with $\tilde{\psi}_x = \tilde{\psi}_0$.

F is a **tempered distribution** if $F \in \mathcal{S}'$. We obtain a natural definition of the Fourier transform F^Λ for a tempered distribution F as

$$F^\Lambda(\varphi) = F(\varphi^\Lambda)$$

for all $\varphi \in \mathcal{S}$. This defines an isomorphism to $\mathcal{S}'$.

Theorem 16.2 (1.7). *For a distribution F supported at the origin,*

$$F = \sum_{|\alpha| \leq N} a_\alpha \partial_x^\alpha \delta$$

is a finite sum.

The **principal value** for a function is the distribution determined by

$$\varphi \mapsto \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} \varphi(x) \frac{dx}{x} = \text{pv} \int_{\mathbb{R}} \varphi(x) \frac{dx}{x}.$$

Then $Hf = \frac{1}{\pi} \text{pv} \left(\frac{1}{x} \right) \star f$ for $f \in \mathcal{S}$, and this distribution $\text{pv} \left(\frac{1}{x} \right) = \frac{d}{dx} (\log |x|) = \frac{1}{2} \left(\frac{1}{x-i0} + \frac{1}{x+i0} \right)$ with Fourier transform $\frac{\pi}{i} \text{sign}(x)$.

A function $f(\mathbb{R}_{\neq 0}^d)$ is **homogeneous of degree** λ if $f_a = f(ax) = a^\lambda f$ for all $a > 0$.

Theorem 16.3 (2.3). *For $-d < \lambda < 0$,*

$$(H_\lambda)^\Lambda = c_\lambda H_{-d-\lambda}, \quad \text{with } c_\lambda = \frac{\Gamma \left(\frac{d+\lambda}{2} \right)}{\Gamma \left(\frac{-\lambda}{2} \right)} \pi^{-\frac{d}{2}-\lambda}.$$

A distribution F is **regular** if there exists a function $f \in C^\infty(\mathbb{R} - \{0\})$ with $F(\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x) dx$ for all $\varphi \in \mathcal{D}(\Omega)$ whose supports are disjoint from the origin. Then f is the function **associated** to F .

Theorem 16.4 (4.2). *The Fourier transform of a regular homogeneous distribution K of degree λ is a regular homogeneous distribution of degree $-d - \lambda$, and vice-versa.*

Theorem 16.5 (2.5). *For a function $f \in C^\infty(\mathbb{R}^d - \{0\})$ homogeneous of degree λ , if there exists m such that $\lambda = -d - m$, there exists unique $F \in (D(\Omega))^*$ of degree λ which agrees with f away from the origin iff*

$$\int_{|x|=1} x^\alpha f(x) d\sigma(x) = 0, \text{ and } F = \sum_{|\alpha|=m} c_\alpha \partial_x^\alpha \delta$$

for all $|\alpha| = m$. If no such m exists, such a unique F exists always, though of different form.

For a partial differential operator

$$L = \sum_{|\alpha| \leq m} a_\alpha \partial_x^\alpha$$

on $\mathbb{C}^d$, a **fundamental solution** of L is $F \in (D(\Omega))^*$ with $L(F) = \mathcal{D}(x)$, the Dirac delta.

Theorem 16.6 (2.8). *For $d \geq 3$, a fundamental solution to $\Delta \in \mathcal{L}(\mathcal{D})$ is*

$$F(x) = -\frac{\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{\frac{d}{2}}} |x|^{-d+2}.$$

Theorem 16.7 (2.9). *For $d = 2$, a fundamental solution to $\Delta \in \mathcal{L}(\mathcal{D})$ is*

$$F(x) = \frac{1}{2\pi} \log |x|.$$

Theorem 16.8 (2.10). *For $L = \frac{\partial}{\partial t} - \Delta_x$, a fundamental solution of L is*

$$F(x, t) = \begin{cases} \mathcal{H}_t(x) & \text{if } t > 0 \\ 0 & \text{else} \end{cases},$$

where $\mathcal{H}_t(x)$ is the heat kernel.

Theorem 16.9 (2.11). *Every linear partial differential equation L on $\mathbb{R}^d$ of constant coefficients has a fundamental solution.*

For a differential operator L of constant coefficients, a **parametrix** for L is a distribution Q with

$$LQ = \delta + r,$$

where the error $r \in \mathcal{S}$.

A partial differential operator $L = \sum_{|\alpha| \leq m} a_\alpha \partial_x^\alpha$ of order m is **elliptic** if its characteristic polynomial P has

$$|P(\xi)| \geq c|\xi|^m$$

for some $c > 0$ for all sufficiently large ξ .

A distribution is **regular** if it agrees with a C^∞ function away from the origin.

Theorem 16.10 (2.12). *Every elliptic operator has a regular parametrix.*

Theorem 16.11 (2.14). *For partial differential operator $L \in \mathcal{L}(\mathcal{D}(\Omega))$ with a regular parametrix, $U \in (\mathcal{D}(\Omega))^*$, $LU = f \in C^\infty(\Omega)$, we have $U \in C^\infty(\Omega)$.*

A $C^{(n)}$ -**normalized bump function** ϕ is a C^∞ function supported in the unit ball with

$$\sup_x |\partial_x^\alpha \phi(x)| \leq 1$$

for all $|\alpha| \leq n$.

We identify **Calderón-Zygmund distributions** as tempered distributions with Fourier transforms $F^\Lambda \in C^\infty$ away from the origin with

$$|\partial^\alpha \xi m(\xi)| \leq c'_\alpha |\xi|^{-\alpha}$$

for all α .

Theorem 16.12 (3.2). *For $T \in \mathcal{L}(\mathcal{S})$, $Tf = f \star K$, where K is a Calderón-Zygmund distribution, T extends to a bounded operator on $L^p(\mathbb{R}^d)$ with $p \in (1, \infty)$.*

Identify **periodic distributions** by $(\mathcal{D}(\mathbb{T}))^*$, the space of C^∞ period functions on $\mathbb{R}^d$, where $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, the d -dimensional torus.

17 Baire Category

For a metric space (X, d) , a set $E \subset X$ is **nowhere dense** if the interior of its closure is empty. We say E is of the **first category** if E is a countable union of nowhere dense sets in X . Otherwise it is a set of the **second category** in X . A set $E \subset X$ is **generic** if its complement is meager (of the first category).

Theorem 17.1 (1.1). *Every complete metric space X is of the second category in itself, that is, X cannot be written as the countable union of nowhere dense sets.*

Theorem 17.2 (2.2). *For $\mathcal{B}$ a Banach space of continuous functions over $[-\pi, \pi]$, the set of continuous functions whose Fourier series diverge on a dense set in $[-\pi, \pi]$ is generic in $\mathcal{B}$.*

18 Besicovitch

Theorem 18.1 (5.1). *The collection of sets in $\mathcal{K}$ of two-dimensional Lebesgue measure zero is generic.*

A **Hamel basis** for a vector space V is a basis of finite dimension.

For a complete metric space (X, d) and a continuous function $T : X \rightarrow X$, an element $x^* \in X$ is **universal** for T if the orbit set $\{T^n(x^*)\}_{n=1}^\infty$ is dense in X .

Define Λ_n to be the set of real numbers x such that there are infinitely many distinct fractions $\frac{p}{q}$ so that

$$\left| x - \frac{p}{q} \right| \leq q^{-n}$$

for every integer $n > 1$. The elements of $\Lambda = \bigcap_{n>1} \Lambda_n$ are the **Liouville numbers**.

19 Probability

The functions $\{r_n\}$ defined by $r_1(t) = \text{sign}(x - \frac{1}{2})$ and

$$r_n(t) = (2^{n-1}t)$$

are the **Rademacher functions**, which are **mutually independent** (i.e., for any sequence of disjoint Borel sets in $\mathbb{R}$ the measure of their intersection is the product of their measures).

Theorem 19.1 (1.4). *For $L = \sum_{n=1}^{\infty} |a_n|^2$, if $L < \infty$, $\sum_{n=1}^{\infty} a_n r_n(t)$ converges a.e. $t \in [0, 1]$, and if L diverges, then $\sum_{n=1}^{\infty} a_n r_n(t)$ diverges a.e. $t \in [0, 1]$.*

Theorem 19.2 (1.7). *For $L = \sum_{n=-\infty}^{\infty} |c_n|^2$, if $L < \infty$, $\rho_n(t) = r_{2n+1}(t)$ if $n \geq 0$, $\rho_n(t) = r_{-2n}(t)$ if $n < 0$,*

$$f_t(\theta) \sim \sum_{n=-\infty}^{\infty} \rho_n(t) c_n e^{in\theta}$$

has $L \in L^p([0, 2\pi])$ for all $p < \infty$. If L diverges, $f_t(\theta)$ is not the Fourier series of an integrable function a.e. $t \in [0, 1]$.

For functions $f_1, \dots, f_N$, their **joint distribution measure** is the measure on $\mathbb{R}^d$ such that for every Borel set $B \subset \mathbb{R}^d$,

$$\mu_{f_1, \dots, f_N}(B) = m(\{x \mid (f_1(x), \dots, f_N(x)) \in B\}).$$

Then sequences $\{f_n\}$ and $\{g_n\}$ of possibly different probability spaces have the **same joint distribution** if

$$\mu_{f_1, \dots, f_N}(B) = \mu_{g_1, \dots, g_N}(B)$$

for all Borel sets $B \subset \mathbb{R}^n$.

A measure τ is **invariant** if $\tau^{-1}(E) = E$ almost everywhere, and it is **ergodic** if there is no proper subset of X , the measure space, which is invariant under τ .

The Fourier transformation of a distribution measure is the measure's **characteristic function**.

A measure is **continuous** if each point has measure zero.

A sequence $\{s_n\}$ of integrable function on X is a **martingale sequence** if

$$s_k = \mathbb{E}_k(s_n)$$

whenever $k \leq n$. A martingale sequence is **complete** if there is an integrable function s_{∞} with $s_k = \mathbb{E}_k(s_{\infty})$ for all k .

Theorem 19.3 (2.18). *(Random walk) If $d < 3$, the random walk is recurrent (i.e., almost all paths return to the origin for infinitely many n). Otherwise, a.e. path returns to the origin at most a finite number of times, and there is a positive probability that the path never returns to the origin.*

The **Walsh-Paley functions** are defined on $[0, 1]$ and denoted by $\{w_n\}$. They are constructed by setting $w_0(t) = 1$, $w_1(t) = r_1(t)$, $w_2(t) = r_2(t)$, and $w_3(t) = r_1(t)r_2(t)$. Then for $n \geq 1$, $n = \sum_{i=1}^{\ell} 2^{k_i}$, where $0 \leq k_1 < \dots < k_{\ell}$, and

$$w_n(t) = \prod_{j=1}^{\ell} r_{k_{j+1}}(t).$$

The dyadic expansion $\alpha \in [0, 1]$ with

$$\alpha = \sum_{j=1}^{\infty} \frac{x_j}{2^j}, \quad x_j \in \{0, 1\},$$

is **normal** if it has a density of 1s equal to the density of 0s. We say a sequence $\{f_n\}_{n=0}^{\infty}$ of functions on X is called a discrete **stationary process** the joint probability distribution of $f_r, f_{r+1}, \dots, f_{r+N}$ is independent of r for every N .

20 Brownian

For a metric (probability) space (Ω, P) and the function $B_t : \Omega \rightarrow \mathbb{R}^d$ defined for all $t \in [0, \infty)$, the **Brownian motion process** $B_t = B_t(\omega)$ has

- 1) $B_0(\omega) = 0$ a.e.
- 2) The increments are independent.
- 3) The increments $B_{t+h} - B_t$ are Gaussian with covariance hI and mean zero for $t \in [0, \infty)$.
- 4) For a.e. $\omega \in \Omega$, the path $t \mapsto B_t(\omega)$ is continuous for $t \in [0, \infty)$.

A measure W on $\mathcal{P}$ (the space of measures of paths with a canonical Ω) is a **Wiener measure** if $\tilde{B}_t(p) = p(t)$ is Brownian.

For every sequence $0 \leq t_1 \leq \dots \leq t_k$ and a Borel set $A \subset \mathbb{R}^{dk}$,

$$\{p \in \mathcal{P} \mid (p(t_1), \dots, p(t_k)) \in A\}$$

is a **cylindrical set**. **Borel measures** are then measures on the Borel subsets of $\mathcal{P}$.

The **section** $\mu^{(t_1, \dots, t_k)}$ of μ is the measure on $\mathbb{R}^{dk}$ given by

$$\mu^{(t_1, \dots, t_k)}(B) = \mu(\{p \in \mathcal{P} \mid (p(t_1), \dots, p(t_k)) \in B\})$$

for Borel measures B .

Theorem 20.1 (3.1). *The measures μ_N on $\mathcal{P}$ converge weakly to the Wiener measure as $N \rightarrow \infty$.*

Theorem 20.2 (4.1).

- 1) $\frac{1}{\sqrt{\delta}} B_{t\delta}$ for every fixed $\delta > 0$,
- 2) $T(B_t)$ for $T \in \mathcal{L}(\mathbb{R}^d)$, and
- 3) $B_{t+\sigma_o} - B_{\sigma_o}$ for $\sigma_o \geq 0$

are all Brownian motion processes.

Theorem 20.3 (4.2). *For the Wiener measure W on $\mathcal{P}$, if $a \in (0, \frac{1}{2})$,*

$$\sup_{\substack{h \in (0, 1] \\ t \in [0, T]}} \frac{|p(t+h) - p(t)|}{h^a} < \infty$$

a.e. p .

For a martingale sequence $\{s_n\}_{n=0}^\infty$ associated to the increasing sequence $\{\mathcal{A}_n\}_{n=0}^\infty$ of σ -algebras on the probability space (X, m) , the function $\tau : x \rightarrow \mathbb{F}$ is a **stopping time** if

$$\{\omega \mid \tau(\omega) \leq t\} \in \mathcal{A}_t$$

for all $n \geq 0$. For a subset A , τ is a **stopping time relative to A** if

$$\{x \in A \mid \tau(x) = n\} \subset A_n$$

for all n .

For the stopping time

$$\tau_\star^x(\omega) = \inf\{t > 0 \mid B_t^x(\omega) \notin \mathcal{R}\},$$

if $\{\tau_\star^x(\omega) = 0\}$ has measure 1, x is a **regular** point at the boundary.

Theorem 20.4 (5.3, Strong Markov). *For a Brownian motion b_t and stopping time σ*

$$B_t^\star(\omega) = B_{t+\sigma(\omega)}(\omega) - B_{\sigma(\omega)}(\omega)$$

is also a Brownian motion independent of $\mathcal{A}_\sigma$.

With $B_t^y(\omega) = y + B_t(\omega)$ and $\tau^y(\omega) = \inf\{t \geq 0 \mid B_t^y(\omega) \notin \mathcal{R}\}$ we have the **stopped process**

$$\hat{B}_t^y(\omega) = y + B_{t \wedge \tau^y(\omega)}(\omega)$$

with $t \wedge \tau^y(\omega) = \min\{t, \tau^y(\omega)\}$.

Theorem 20.5 (5.5). *For σ and τ stopping times with $\sigma(\omega) \leq \tau(\omega)$ and bounded Borel function F on $\mathbb{R}^d$,*

$$\int_{\Omega} F\left(\hat{B}_{\sigma(\omega)+t}(\omega)\right) dP(\omega) = \int \int_{\Omega \times \Omega} F\left(\hat{B}_t^{y(\omega')}(\omega)\right) dP(\omega) dP(\omega')$$

for all $t \geq 0$ and $y(\omega') = \hat{B}_{\sigma(\omega')}(\omega')$. Alternatively,

$$\mathbb{E}_{\mathcal{A}_\sigma}(f(B_{\sigma(\omega)+\gamma})) = \int_{\Omega} f(B_\gamma(\omega) + x) dP(\omega) \Big|_{x=B_{\sigma(\omega')}(\omega')},$$

and

$$\int_A F\left(\hat{B}_{\sigma(\omega)+t}(\omega)\right) dP(\omega) = \int_A \int_{\Omega} F\left(\hat{B}_t^{y(\omega')}(\omega)\right) dP(\omega) dP(\omega')$$

whenever $A \in \mathcal{A}_\sigma$.

Theorem 20.6 (6.1). *For*

$$u(x) = \int_{\partial \mathcal{R}} f(y) d\mu^x(y),$$

u is harmonic and $u(x) \rightarrow f(y)$ as $x \rightarrow y$ if $x \in \mathcal{R}$ and y is a regular point of $\partial \mathcal{R}$.

A **truncated cone** Γ is the open set

$$\Gamma = \{y \in \mathbb{R}^d \mid |y| < \alpha(y \cdot \gamma), |y| < \delta\}$$

for γ a unit vector, $\alpha > 1$ and $\delta > 0$ fixed.

A bounded open set $\mathcal{R}$ satisfies the **outside cone condition** if whenever $x \in \partial \mathcal{R}$ there exists a truncated cone Γ so that $x + \Gamma$ is disjoint from $\mathcal{R}$.

Corollary 20.7 (6.3, Dirichlet Analogue). *If $\mathcal{R}$ satisfies the outside cone condition, and f is a continuous function on $\partial\mathcal{R}$, there exists a unique function u which is continuous in $\overline{\mathcal{R}}$ and harmonic in $\mathcal{R}$ with $u|_{\partial\mathcal{R}} = f$.*

A bounded open set $\mathcal{R}$ has a **Lipschitz boundary** if $\partial\mathcal{R}$ can be covered by finitely many balls so that for each such ball B , the set $\partial\mathcal{R} \cap B$ can be written as $x_d = \varphi(x_1, \dots, x_{d-1})$ where φ is a function that satisfies a Lipschitz condition.

21 Several Complex Variables

$\mathbb{P}_r(z^0)$ denotes the **polydisc**

$$\mathbb{P}_r(z^0) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j - z_j^0| < r_j \text{ for all } j \in [0, n]\},$$

while $C_r(z^0)$ denotes the product of boundary circles

$$C_r(z^0) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j - z_j^0| = r_j, j \in [0, n]\},$$

and z^α denotes the monomial

$$\prod_{i=1}^n z_i^{\alpha_i}$$

where $\alpha_i \in \mathbb{N}$.

The **Cauchy–Riemann equations** are

$$\frac{\partial f}{\partial \bar{z}_j} = 0$$

where

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right), \text{ and } z_j = x_j + iy_j,$$

$x_j, y_j \in \mathbb{R}$.

Theorem 21.1 (3.3). *For $\Omega \subset \mathbb{C}^{n \geq 2}$ bounded and $K \subset \Omega$ compact with $\Omega - K$ connected, any function F_0 analytic in $\Omega - K$ has an analytic continuation into Ω .*

For $\Omega \subset \mathbb{R}^d$, a defining function $\rho : \mathbb{R}^d \rightarrow \mathbb{R}$ of Ω has

$$\begin{cases} \rho(x) < 0, & \text{when } x \in \Omega, \\ \rho(x) = 0, & \text{when } x \in \partial\Omega, \\ \rho(x) > 0, & \text{when } x \in \overline{\Omega}^c \end{cases}$$

The boundary of Ω is of **class** C^k if Ω has a defining function ρ with

$$\rho \in C^k(\mathbb{R}^d) \text{ and } |\nabla \rho(x)| > 0$$

if $x \in \partial\Omega$.

M is a **hypersurface** of class C^k if there exists $\rho : B \subset \mathbb{R}^d \rightarrow \mathbb{R}$ of class C^k with

$$M = \{x \in B \mid \rho(x) = 0\} \quad \text{and} \quad |\nabla \rho(x)| > 0$$

if $x \in M$.

A function f_0 defined on $\partial\Omega$ is of **class** C^ℓ if there is an extension f of f_0 to $\mathbb{R}^d$ so that f is class C^ℓ on $\mathbb{R}^d$.

Vector fields on $\mathbb{C}^n$ expressed as

$$\sum_{j=1}^n \left(a_j(z) \frac{\partial}{\partial \bar{z}_j} + b_j(z) \frac{\partial}{\partial z_j} \right)$$

are **Cauchy–Riemann vector fields** (i.e., annihilate all holomorphic functions). Such vector fields are **tangential** if

$$\sum_{j=1}^n a_j(z) \rho_j(z) = 0,$$

where $\rho_j(z) = \frac{\partial \rho}{\partial \bar{z}_j}$.

Theorem 21.2 (4.1, Bochner). *For $\Omega \subset \mathbb{C}^n$ with boundary of class C^3 and $\bar{C}^c$ connected, if F_0 is a function of class C^3 on $\partial\Omega$ satisfying the tangential Cauchy–Riemann equations there exists a holomorphic function F on Ω which is continuous on $\bar{\Omega}$ such that $F|_{\partial\Omega} = F_0$.*

The **Levi form** of Ω at a boundary point z_0 is the Hermitian matrix

$$\left\{ \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(z_0) \right\}_{1 \leq j, k \leq n-1}.$$

The boundary point z_0 is **pseudoconvex** if the Levi form is positive semidefinite, and **strongly pseudoconvex** if the Levi form is positive definite.

A continuous function f satisfies the tangential Cauchy–Riemann equations in the **weak sense** if

$$\int_{\mathbb{C}^{n-1} \times \mathbb{R}} f L_j^t(\psi) dz' dx = 0$$

for all $\psi \in C^1$ with sufficiently small supports, and

$$L_j(f) = \frac{\partial f}{\partial \bar{z}_j} - a_j \frac{\partial f}{\partial \bar{z}_n},$$

where $a_j = \frac{\rho_j}{\rho_n}$, and

$$L_j^t(\psi) = - \left(\frac{\partial \psi}{\partial \bar{z}_j} - \frac{1}{2} \cdot \frac{\partial(a_j \psi)}{\partial x_n} \right).$$

Theorem 21.3 (7.1). *For $M \subset \mathbb{C}^n$ a hypersurface of class C^2 defined near $z^0 \in M$, with z_0 having been brought to the origin after a complex affine-linear change of coordinates thus representing M by*

$$M = \{z = (z', z_n) \mid \text{Im}(z_n) = \varphi(z', x_n)\},$$

there exist open balls B'_{z_0} and B_{z_0} with $\bar{B}' \subset B$ such that if $F : M \cap B \rightarrow \mathbb{C}$ satisfies the tangential Cauchy–Riemann equations in the weak sense, then F can be uniformly approximated on $M \cap \bar{B}'$ by polynomials in $z_1, \dots, z_n$.

Theorem 21.4 (7.5). *For a Levi form with at least one strictly positive eigen value for each $z \in M$, for every $z^0 \in M$ there exists a ball B'_{z_0} such that if $F_0 : M \rightarrow \mathbb{C}$ is a continuous function on M which satisfies the tangential Cauchy–Riemann equations in the weak sense, there exists an f which is holomorphic in $\Omega_- \cap B'$ and continuous in $\bar{\Omega}_- \cap B'$ with $F(z) = F_0(z)$ for $z \in M \cap B'$.*

22 Upper half-space

Define the **upper half-space** $\mathcal{U} \subset \mathbb{C}^n$ by

$$\mathcal{U} = \{z \in \mathbb{C}^n \mid \operatorname{Im}(z_n) > |z'|^2\}$$

with $z = (z', z_n)$ and $z' = (z_1, \dots, z_{n-1})$. Note that $\mathcal{U}$ is holomorphically equivalent with the unit ball via a fractional linear transformation

$$w_n = \frac{i - z_n}{i + z_n}, \quad w_k = \frac{2iz_k}{i + z_n}, \quad k \in [n-1].$$

The Hardy space $H^2(\mathcal{U})$ is then every holomorphic function in $\mathcal{U}$ F with

$$\sup_{\epsilon > 0} \int_{\partial \mathcal{U}} |F(z', z_n + i\epsilon)|^2 d\beta < \infty.$$

It is also a Hilbert space.

Define the Cauchy integral in $\mathcal{U}$ by setting for each $z, w \in \mathbb{C}^n$

$$r(z, w) = \frac{i}{2}(\bar{w}_n - z_n) - z' \cdot \bar{w}'$$

as well as

$$S(z, w) = c_n r(z, w)^{-n}$$

where $c_n = \frac{(n-1)!}{(4\pi)^n}$. Then the Cauchy integral $C(f)$ is

$$C(f)(z) = \int_{\partial \mathcal{U}} S(z, w) f(w) d\beta(w)$$

for $z \in \mathcal{U}$.

Theorem 22.1 (8.7). *For U a distribution defined on $\mathbb{C} \times \mathbb{R}$ with $\bar{L}(U) = f$ in a neighborhood of the origin, $C(f)(z)$ must have an analytic continuation to a neighborhood of 0.*

It then follows that there exist non-solvable differential equations $\bar{L}(U) = f$ near the origin though $f \in C^\infty$.

A **domain of holomorphy** is a region Ω such that there exists a holomorphic function F so that for every $z^0 \in \partial\Omega$, F cannot be continued into some ball centered at z^0 .

Theorem 22.2 (Problem 5). *For Ω a domain of holomorphy with boundary of class C^2 , Ω is psuedoconvex. Further, if Ω is psuedoconvex, then it is a domain of holomorphy.*

Theorem 22.3 (Problem 6). *For f a distribution of compact support with $C(f)(z)$ analytic near $z = 0$, $\bar{L}(U) = f$ is locally solvable near the origin.*

23 Oscillatory Integrals

The **averaging operator** A is defined by

$$A(f)(x) = \frac{1}{\sigma(S^{d-1})} \int_{S^{d-1}} f(x - y) d\sigma(y)$$

with $d\sigma$ the induced measure on S^{d-1} .

An **oscillatory integral** $I(\lambda)$ has form

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(x)} \psi(x) dx,$$

where Φ is the **phase** and ψ the **amplitude**.

For the representation of a C^{d+2} hypersurface M by

$$M = \{(x', x_d) \in \tilde{B} \mid x_d = \varphi(x')\}$$

where $\tilde{B}$ is a ball centered at the origin and $\varphi \in C^\infty$ self-affine with $\nabla_{x'} \varphi(x')|_{x'=0}$ and defining function $\rho_1(x) = \varphi(x') - x_d$ of canonical normalization $|\nabla \rho| = 1$, the **curvature (second fundamental) form** of M at $x \in M$ is

$$\sum_{i \leq k, j \leq d} \xi_k \xi_j \frac{\partial^2 \rho}{\partial x_k \partial x_j}(x),$$

defined only for vectors $\sum \xi_k \frac{\partial}{\partial x_k}$ tangent to M at x . Via rotation in $x' \in \mathbb{R}^{d-1}$ space and relabeling of coordinates

$$\varphi(x') = \frac{1}{2} \sum_{j=1}^{d-1} \lambda_j x_j^2 + O(|x'|^3)$$

for which the eigenvalues λ_j are the **principle curvatures** of M at x^0 and the determinant the **total (Gauss) curvature** of M . Then with the induced Lebesgue measure on M $d\sigma$, for any continuous function f on M of compact support with continuous extension F into a neighborhood of M ,

$$\int_M f d\sigma = \int_{\mathbb{R}^{d-1}} f(x', \varphi(x')) (1 + |\nabla_{x'} \varphi|^2)^{\frac{1}{2}} dx',$$

where $d\mu$ is a **surface-carried measure** on M with **smooth density** if $d\mu = \psi d\sigma$ with $\psi \in C^\infty$ and of compact support.

Theorem 23.1 (3.1). *For a hypersurface M of non-vanishing Gauss curvature at each point of the support of $d\mu$,*

$$|\widehat{d\mu}(\xi)| \in O\left(|\xi|^{-\frac{(d-1)}{2}}\right)$$

as $|\xi| \rightarrow \infty$.

Theorem 23.2 (4.1). *For $d\mu$ with Gauss curvature non-vanishing at each point $x \in M$ in the support of $d\mu$,*

$$A : L^2(\mathbb{R}^d) \rightarrow L^2_{\frac{d-1}{2}}(\mathbb{R}^d),$$

and A extends to some T with

$$T \in \mathcal{L}(L^{\frac{d+1}{d}}(\mathbb{R}^d), L^{d+1}(\mathbb{R}^d)).$$

Define an **analytic family of operators** by a linear mapping $T_s : L_{\text{simple}}(\mathbb{R}^d) \rightarrow L_{\text{loc}}(\mathbb{R}^d)$ with

$$\Phi_0(s) = \int_{\mathbb{R}^d} T_s(f) g dx$$

continuous and bounded in S and analytic in the interior of S for any simple functions f and g for every $s \in S = \{\operatorname{Re}(s) \in [a, b]\}$, and

$$\sup_{t \in \mathbb{R}} \|T_{a+it}(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}, \quad \sup_{t \in \mathbb{R}} \|T_{b+it}(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}.$$

The (L^p, L^q) **restriction** is defined by

$$\left(\int_M |\hat{f}(\xi)|^q d\mu(\xi) \right)^{\frac{1}{q}} \leq c \|f\|_{L^p(\mathbb{R}^d)}.$$

Theorem 23.3 (5.2). *For local hypersurface M of nonzero Gauss curvature at each point in the support of $d\mu$, the*

$$\left(L^{\frac{2d+2}{d+3}}, L^2 \right)$$

restriction holds.

Theorem 23.4 (6.3, **Strichartz estimate**). *For $u(x, t) = e^{it\Delta}(f)(x)$, $f \in L^2(\mathbb{R}^d)$,*

$$\|u(x, t)\|_{L^{\frac{2d+4}{d}}(\mathbb{R}^d \times \mathbb{R})} \leq c \|f\|_{L^2(\mathbb{R}^d)}.$$

Theorem 23.5 (6.4). *The solution $e^{t(\frac{d}{dx})^3}(f)$ satisfies*

$$\|u\|_{L^8(\mathbb{R}^2)} \leq c \|f\|_{L^2(\mathbb{R})}.$$

Define the **solution operator** S for the non-homogenous Schrodinger equation $\frac{1}{i} \frac{\partial u}{\partial t} - \Delta u = F$ with $u(x, 0) = 0$ by

$$S(f)(x, t) = i \int_0^t e^{i(t-s)\Delta} F(\cdot, s) ds.$$

Theorem 23.6 (6.7).

$$\|S(F)\|_{L^{\frac{2d+4}{d}}(\mathbb{R}^d \times \mathbb{R})} \leq c \|F\|_{L^{\frac{2d+4}{d+4}}(\mathbb{R}^d \times \mathbb{R})}.$$

The **rotational matrix** $\mathcal{M}$ of $\rho \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is given by

$$\mathcal{M} = \begin{pmatrix} \rho & \frac{\partial \rho}{\partial y_1} & \cdots & \frac{\partial \rho}{\partial y_d} \\ \frac{\partial \rho}{\partial x_1} & \ddots & & \\ \vdots & & \frac{\partial^2 \rho}{\partial y_j \partial x_k} & \\ \frac{\partial \rho}{\partial x_d} & & & \end{pmatrix},$$

and the **rotational curvature** $\operatorname{rotcurv}(\rho)$ by $\det \mathcal{M}$.

24 Lattice Points

Let $r_2(k)$ be the number of representation of k as the sum of two squares and $d(k)$ the number of divisors of k . Identify immediately that

$$\sum_{k=1}^{\mu} r_2(k) = \pi\mu + O(\sqrt{\mu}) \quad \text{and} \quad \sum_{k=1}^{\mu} d(k) = \mu \log \mu + 2(\gamma - 1)\mu + O(\sqrt{\mu})$$

as $\mu \rightarrow \infty$.

A set Ω is **strongly convex** when the quadratic (second fundamental) form is strictly positive definite at each point of the boundary of Ω .

Theorem 24.1 (8.4). *For $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary and strongly convex with $0 \in \Omega$,*

$$N_R = R^d m(\Omega) + O\left(R^{d-\frac{2d}{d+1}}\right)$$

as $R \rightarrow \infty$, where N_R counts the amount of lattice points in $\{Rx \mid x \in \Omega\}$.

Define the **hyperbolic measure** on $\mathbb{R}^2$ d by

$$(E) = \int_0^\infty \chi_E\left(u, \frac{1}{u}\right) \frac{1}{u} du$$

for every Borel set $E \subset \mathbb{R}^2$.

Theorem 24.2 (8.5).

$$\sum_{1 \leq k \leq \mu} d(k) = \mu \log \mu + (2\gamma - 1)\mu + O\left(\mu^{\frac{1}{3}} \log \mu\right).$$

For a local hypersurface M in $\mathbb{R}^d$ with a smooth vector field ν for the neighborhood of a point $x_0 \in M$, the **Gauss map** $x \mapsto v(x)$ is a unit normal vector of M at each $x \in M$.

References

- [Axl24] Sheldon Axler. *Linear Algebra Done Right*. 4th ed. Springer, 2024.
- [Rud76] Walter Rudin. *Principles of Mathematical Analysis*. 3rd ed. McGraw-Hill, 1976.
- [SS03a] Elias M. Stein and Rami Shakarchi. *Complex Analysis*. Vol. 2. Princeton Lectures in Analysis. Princeton, NJ: Princeton University Press, 2003.
- [SS03b] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Vol. 1. Princeton Lectures in Analysis. Princeton, NJ: Princeton University Press, 2003.
- [SS05] Elias M. Stein and Rami Shakarchi. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Vol. 3. Princeton Lectures in Analysis. Princeton, NJ: Princeton University Press, 2005.
- [SS11] Elias M. Stein and Rami Shakarchi. *Functional Analysis: Introduction to Further Topics in Analysis*. Vol. 4. Princeton Lectures in Analysis. Princeton, NJ: Princeton University Press, 2011.