Calculus

Demarri Dosunmu

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1 Distance

For a plane $S = Ax + By + Cz - k = 0$, reformulate it as $\mathbf{n} \cdot \mathbf{x} = k$. For some point $\mathbf{p} \notin S$, consider the direct path $\mathbf{p} - k\langle \frac{1}{A}, \frac{1}{B}, \frac{1}{C} \rangle$, and remove the displacement inside the plane:

$$\mathbf{b} = \mathbf{p} - k\langle \frac{1}{A}, \frac{1}{B}, \frac{1}{C} \rangle, \qquad d = \|\frac{\mathbf{n} \cdot \mathbf{b}}{\mathbf{n} \cdot \mathbf{n}}\|.$$

This is the **projection** of the geodesic $\mathbf{b}$ onto the orthogonal complement of the plane.

2 Implicit Function

For the gradient of a function $f : D \to \mathbb{R}$ excepting one element at $\mathbf{x} \in D$ there exists a unique candidate for the unknown element for a small enough neighborhood centered at $\mathbf{x}$.

3 Lagrange

Let X be open in $\mathbb{R}^n$ and $f, g : X \to \mathbb{R}$ be differentiable. Let $S = \{\mathbf{x} \in X \mid g(\mathbf{x}) = c\}$. Then if $f(X)$ has an extremum point at $\mathbf{x}_0 \in S$ such that $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, $\exists \lambda$ so

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0).$$

Generally, for $f, g_1, \ldots, g_k : X \subseteq \mathbb{R}^n \to \mathbb{R}$ differentiable and $k < n$, $S = \{\mathbf{x} \in X \mid g_1(\mathbf{x}) = c_1, \ldots, g_k(\mathbf{x}) = c_k\}$, if $f(S)$ has an extremum point at $\mathbf{x}_0$ and $(\nabla g_1(\mathbf{x}_0), \ldots, \nabla g_k(\mathbf{x}_0))$ are linearly independent vectors, then $\exists\ (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ with

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \lambda_2 \nabla g_2(\mathbf{x}_0) + \cdots + \lambda_k \nabla g_k(\mathbf{x}_0).$$

4 Lines

1. they have arclength parameterization:

$$\mathbf{s}(l) = \mathbf{x}(\mu) \qquad \Big(\int_a^\mu \|\mathbf{x}'(t)\| \, dt = l \Big)$$

2. there are equilibrium points in vector fields:

$$D\mathbf{F} = \mathbf{0}$$

3. there exists curvature:

$$\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

5 Theorems

5.1 Green

$$\oint_C M\ dx + N\ dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy$$

i.e.,

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

5.1.1 Divergence under Green

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA$$

5.2 Stokes

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

5.3 Gauss

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

5.3.1 Formulae

$$\iiint_D \nabla f \cdot \nabla g\ dV + \iiint_D f \nabla^2 g\ dV = \oiint_S f \nabla g \cdot d\mathbf{S}$$

$$\iiint_D \nabla g \cdot \nabla f\ dV + \iiint_D g \nabla^2 f\ dV = \oiint_S g \nabla f \cdot d\mathbf{S}$$

If

$$f(\mathbf{r}) = -\frac{1}{4\pi} \iiint_D \frac{\nabla^2 f(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|}\, dV + \frac{1}{4\pi} \oiint_S \left(-\frac{f(\mathbf{x})}{\nabla \|\mathbf{r} - \mathbf{x}\|} + \frac{\nabla f(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|}\right) \cdot d\mathbf{S},$$

then for $\varphi = \nabla^2 f$,

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi} \nabla^2_{\mathbf{r}} \iiint_D \frac{\varphi(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|}\, dV.$$

5.4 Maxwell

$$\text{Flux of } \mathbf{E} = \oiint_S \mathbf{E} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{E}\, dV$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

6 Differential Forms

6.1 0-forms

Any differentiable scalar function f is a **differential 0-form**.

6.2 1-forms

A **differential 1-form** ω is an expression that is built from the basic 1-forms as

$$\omega = F_1(x_1, \dots, x_n)\ dx_1 + F_2(x_1, \dots, x_n)\ dx_2 + \cdots + F_n(x_1, \dots, x_n)\ dx_n$$

where F_j is a k-times differentiable scalar function over the domain U. Then the product between a 0-form f and a 1-form ω is defined as

$$f\omega = fF_1\ dx_1 + fF_2\ dx_2 + \cdots + fF_n\ dx_n.$$

6.2.1 Use

1-forms may be used as functions and are equivalent to the dual basis of the vector space. For basic 1-forms $(dx_1, \dots, dx_n)$ the argument vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{F}^n$ (for a field $\mathbb{F}$) so

$$dx_i(\mathbf{a}) = a_i.$$

Hence for any $\mathbf{x} \in D$, we have

$$\omega_{\mathbf{x}} = F_1(\mathbf{x})dx_1 + \cdots + F_n(\mathbf{x})dx_n$$

where we may also specify another vector $\mathbf{a} \in \mathbb{F}^n$:

$$\omega_{\mathbf{x}}(\mathbf{a}) = F_1(\mathbf{x})dx_1(\mathbf{a}) = F_2(\mathbf{x})dx_2(\mathbf{a}) + \cdots + F_n(\mathbf{x})dx_n(\mathbf{a})$$

which (canonically) equals

$$F_1(\mathbf{x})a_1 + \cdots + F_n(\mathbf{x})a_n.$$

Note that 1-forms are functions on vector fields (specify a location and direction).

6.3 2-forms

A basic **differential 2-form** $\varphi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ is an expression of the form

$$dx_i \wedge dx_j, \qquad (i, j) \in [n]^2$$

where

$$(dx_i \wedge dx_j)(\mathbf{a}, \mathbf{b}) = \det \begin{bmatrix} dx_i(\mathbf{a}) & dx_i(\mathbf{b}) \\ dx_j(\mathbf{a}) & dx_j(\mathbf{b}) \end{bmatrix}.$$

Such a scalar geometrically represents the area of the parallelogram spanned by the projections of $\mathbf{a}$ and $\mathbf{b}$ in the x_ix_j-plane. The definition implies

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad dx_i \wedge dx_i = 0$$

making such forms alternating bilinear forms on $\mathbb{F}^n$ (see Shaxler 9A). Hence for $\mathbf{x} = (x_1, x_2, \dots, x_n)$, a general differential 2-form on $U \subseteq \mathbb{F}^n$ is an expression

$$\omega = F_{1,2}(\mathbf{x})\ (dx_1 \wedge dx_2) + F_{1,3}(\mathbf{x})\ (dx_1 \wedge dx_3) + \cdots + F_{n-1,n}(\mathbf{x})\ (dx_{n-1} \wedge dx_n).$$

Note that this resembles the generalization of integration with respect to two variables.

6.4 k-forms

For $k \in \mathbb{Z}_+$, a basic **differential k-form** on $\mathbb{F}^n$ is an expression of the form

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where $i \leq i_j \leq n$ for $j \in [k]$. The basic k-forms are functions which require k vector arguments $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k) \in \mathbb{F}^{k,k}$ so that

$$dx_{i_i} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = \det \begin{bmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \cdots & dx_{i_1}(\mathbf{a}_k) \\ dx_{i_2}(\mathbf{a}_1) & dx_{i_2}(\mathbf{a}_2) & \cdots & dx_{i_2}(\mathbf{a}_k) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \cdots & dx_{i_k}(\mathbf{a}_k) \end{bmatrix}.$$

6.5 Alternative Formulation

A general differential k-form on $U \subseteq \mathbb{F}^n$ is an expression of the form

$$\omega = \sum_{i_i, \ldots, i_k = 1}^{n} F_{i_1, \ldots, i_k}(\mathbf{x}) \; (dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

where every $F_{i_1, \ldots, i_k}$ is a function from U to $\mathbb{F}$. Then for any $\mathbf{x} \in U$ and a vector of vectors $(\mathbf{a}_1, \ldots, \mathbf{a}_k) \in \mathbb{F}^{k,k}$,

$$\omega_{\mathbf{x}}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = \sum_{i_1, \ldots, i_k = 1}^{n} F_{i_1, \ldots, i_k}(\mathbf{x})(dx_{i_1} \wedge \cdots \wedge dx_{i_k})(\mathbf{a}_1, \ldots, \mathbf{a}_k).$$

Note that k in k-form connotes the amount of vector inputs.

6.6 Exterior Product

We have effectively provided an account of the **exterior (or so-called "wedge") product** denoted by $\wedge$. Let $U \subset \mathbb{F}^n$ be open. For f a 0-form on U, $\omega = \sum F_{i_1, \ldots, i_k}(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$ a k-form on U and $\eta = \sum G_{j_1, \ldots, j_l}(dx_{j_1} \wedge \cdots \wedge dx_{j_l})$ an l-form,

$$f \wedge \omega = f\omega = \sum f F_{i_1, \ldots, i_k}(dx_{i_1} \wedge \cdots \wedge dx_{i_k}), \text{ and}$$

$$f \wedge \omega \wedge \eta = \sum f F_{i_1, \ldots, i_k} G_{j_i, \ldots, j_l}(dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}),$$

making the wedge product of a k-form and l-form a $(k+l)$-form.

6.7 Integrals

6.7.1 Line

For $\mathbf{x} : [a, b] \to \mathbb{F}^n$ a C^1 path, if ω is a 1-form with image $R \subset U$ for open $U \subseteq \mathbb{F}^n$, then the integral of ω over $\mathbf{x}$ is

$$\int_{\mathbf{x}} \omega = \int_a^b \omega_{\mathbf{x}(t)}(\mathbf{x}'(t)) \; dt,$$

which clearly fits for vector field line integrations.

6.7.2 Surface

For a bounded and connected region D in $\mathbb{F}^2$ and $\mathbf{X} : D \to \mathbb{F}^3$ a smooth parametrized surface, if ω is a 2-form defined over an open set in $\mathbb{F}^3$ containing $\mathbf{X}(D)$, then the integral of ω over $\mathbf{X}$ is

$$\int_{\mathbf{X}} \omega = \iint_D \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t)\, ds\, dt, \qquad \mathbf{T}_s = \frac{\partial \mathbf{X}}{\partial s},\ \mathbf{T}_t = \frac{\partial \mathbf{X}}{\partial t},$$

which clearly fits for surface integrals.

7 Manifolds

For a region in $D \subset \mathbb{F}^n$ open and connected, a parametrized k-manifold in $\mathbb{F}^n$ is a continuous map $\mathbf{X} : D \to \mathbb{F}^n$ for which injectivity is exigible only for ∂D. Then the image $M = \mathbf{X}(D)$ is the **underlying manifold** of $\mathbf{X}$. This k-manifold possesses k **coordinate curves** defined from $\mathbf{X}$ by currying each $u_1, \dots, u_k$—the j-th coordinate curve is the curve parametrized by

$$u_j \mapsto \mathbf{X}(a_1, \dots, a_{j-1}, u_j, a_{j+1} \dots, a_k)$$

for which a_i are constants. Then if $\mathbf{X}$ is differentiable and $x_1, x_2, \dots, x_n$ denote the component functions of $\mathbf{X}$, the **tangent vector** to the j-th coordinate curve $\mathbf{T}_{u_j}$ is

$$\mathbf{T}_{u_j} = \frac{\partial \mathbf{X}}{\partial u_j} = \left(\frac{\partial x_1}{\partial u_j}, \frac{\partial x_2}{\partial u_j}, \cdots, \frac{\partial x_n}{\partial u_j} \right).$$

We say that a parametrized k-manifold is **smooth** at a point $\mathbf{X}(\mathbf{u})$ if the mapping $\mathbf{X}$ is class C^1 in a ball centered at $\mathbf{u}$ and the k tangent vectors are linearly independent at $\mathbf{X}(\mathbf{u})$. Hence the entire parametrized k-manifold is smooth if it is smooth at every interior point of D.

7.1 Integrals

For a bounded and connected region $D \subset \mathbb{F}^n$ and $\mathbf{X} : D \to \mathbb{F}^n$ a smooth parametrized k-manifold, if ω is a k-form defined on an open set in $\mathbb{F}^n$ which contains $M = \mathbf{X}(D)$, then the integral of ω over M is

$$\int_{\mathbf{X}} \omega = \int \dots \int_D \omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \dots, \mathbf{T}_{u_k})\, du_1 \cdots du_k.$$

7.2 Orientation

Note that it is nontrivial to show that the parametrization of the manifold does not affect the integral over the underlying manifold.

7.2.1 Orientable, Oriented

For $\mathbf{X} : D \subseteq \mathbb{F}^k \to \mathbb{F}^n$ and $M = \mathbf{X}(D)$ a smooth parametrized k-manifold, an **orientation** of M is a choice of smooth and nonzero k-form Ω defined on M. If such a k-form exists, M is said to be **orientable** and thence **oriented** once such a choice of a k-form is made.

7.2.2 Compatible

For M oriented by the k-form Ω, the tangent vectors $\mathbf{T}_{u_1}, \dots, \mathbf{T}_{u_k}$ for the coordinate curves of M are said to be **compatible** with Ω if

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \dots, \mathbf{T}_{u_k}) > 0 \qquad \forall \mathbf{u} \in D,$$

and we say that the parametrization $\mathbf{X}$ is compatible with the orientation of Ω if the resulting tangent vectors $\mathbf{T}_{u_1}, \dots, \mathbf{T}_{u_k}$ are.

7.2.3 Consistently Oriented

For a smooth parametrized k-manifold $M \subset \mathbb{F}^n$ with boundary ∂M, if M is oriented by the k-form Ω, then the connected pieces of ∂M are said to be **oriented consistently** with M, or that ∂M has its orientation **induced** from that of M if the orientation of the $(k-1)$-form $\Omega^{\partial M}$ is determined from Ω by the following:

Let $\mathbf{V}$ be the unique outward-pointing unit vector in $\mathbb{F}^n$ varying continuously over ∂M that is tangent to M and normal to ∂M. Then $\Omega^{\partial M}$ is defined as

$$\Omega^{\partial M}_{\mathbf{Y}(\mathbf{s})}(\mathbf{a}_1, \dots, \mathbf{a}_{k-1}) = \Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{V}, \mathbf{a}_1, \dots, \mathbf{a}_{k-1})$$

where $\mathbf{X} : D \subseteq \mathbb{F}^k \to \mathbb{F}^n$ parametrizes M, the map $\mathbf{Y} : E \subseteq \mathbb{F}^{k-1} \to \mathbb{R}^n$ parametrizes a connected piece of ∂M, and $\mathbf{Y}(\mathbf{s}) = \mathbf{X}(\mathbf{u})$. This entails that $\mathbf{V}$ is tangent to M, $\mathbf{V}$ is normal to ∂M, and $\mathbf{V}$ "points away" from M.

8 Generalized Stokes

8.1 Exterior Derivative

The **exterior derivative** df of a 0-form f on $U \subseteq \mathbb{F}^n$ is the 1-form

$$df = \frac{\partial f}{\partial x_1}\, dx_1 + \frac{\partial f}{\partial x_2}\, dx_2 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n.$$

For $k > 0$, the exterior derivative of some k-form $\omega = \sum F_{i_1,\dots,i_k}\ (dx_{i_1} \wedge \cdots \wedge dx_{i_k})$ is the $(k+1)$-form

$$d\omega = \sum (dF_{i_1,\dots,i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $dF_{i_1,\dots,i_k}$ is computed as the exterior derivative of a 0-form.

8.2 Generalize

For $D \subseteq \mathbb{F}^k$ be a closed, bounded, and connected, $M = \mathbf{X}(D)$ an oriented parametrized k-manifold in $\mathbb{F}^n$, $\partial M \neq 0$, ∂M having orientation induced from that of M, ω a $(k-1)$-form defined on an open set in $\mathbb{F}^n$ containing M,

$$\int_M d\omega = \int_{\partial M} \omega.$$

If $\partial M = \emptyset$, then we take $\int_{\partial M} \omega$ to be 0 in the preceding equation.

9 Further Reading

James R. Munkres, Analysis on Manifolds, Addison-Wesley, 1991.