

Partial Differential Equations (I & II)

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1 Introduction

A **partial differential equation (PDE) of order** $k \in \mathbb{N}$ is an expression of the form

$$F(x, u(x), Du(x), \dots, D^k u(x)) = 0,$$

where $x \in \Omega \subseteq \mathbb{R}^n$ open, $u : \Omega \rightarrow \mathbb{R}$, and $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \dots \times \mathbb{R}^{r^k} \rightarrow \mathbb{R}$. A **linear partial differential equation** has form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

and is **homogeneous** if $f \equiv 0$. A **semi-linear partial differential equation** has form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

A **quasi-linear partial differential equation** has form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0.$$

A k^{th} -order system of partial differential equations has form

$$\mathbf{F} \left(D^k \mathbf{u}(x), D^{k-1} \mathbf{u}(x), \dots, D \mathbf{u}(x), \mathbf{u}(x), x \right) = \mathbf{0}$$

with unknown $\mathbf{u} : U \rightarrow \mathbb{R}^m$.

2 Quintessential Linear PDEs

2.1 Transport

The **transport equation** with constant coefficients is modeled by

$$u_t + b \cdot Du = 0,$$

with $b \in \mathbb{R}^n, u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$, and $u = u(x, t)$. Notice that for a reparameterization of u by a single variable we can identify a function with derivative 0 in the reparameterization. For an **initial-value** problem version, where $u = g$ for $(x, t) \in \mathbb{R}^n \times \{t = 0\}$, we deduce

$$u(x, t) = g(x - tb),$$

the weak solution (g is not necessarily differentiable). In the non-homogeneous case

$$u_t + b \cdot Du = f,$$

the same reparameterization, algebra, and elementary calculus yield

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s) ds.$$

2.2 Laplace

Laplace's equation is modeled by

$$\Delta u = 0$$

and **Poisson's** is modeled by

$$-\Delta u = f.$$

The fundamental solution of Laplace's equation is given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}|x| & (n = 2) \\ \frac{1}{n(n-2)m(B_1)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases}.$$

Theorem 2.1 (1). For $u(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y) dy$,

$$-\Delta u = f.$$

Theorem 2.2 (2). For $\Delta u = 0$,

$$u(x) = \frac{1}{m(\partial B_r(x))} \int_{\partial B_r(x)} u dS = \frac{1}{m(B_r(x))} \int_{B_r(x)} u dy$$

for every ball $B_r(x)$.

Theorem 2.3 (4, (Strong) Maximum Principle). For $u \in C^2(U) \cap C(\overline{U})$ harmonic,

$$\max_{\overline{U}} u = \max_{\partial U} u,$$

and (strongly) if U connected, there exists $x_0 \in U$ such that

$$u(x_0) = \max_{\overline{U}} u \implies Du = 0$$

in U .

The **(Strong) Minimum Principle** is the above with max replaced by min and u by $-u$.

Theorem 2.4 (5, Uniqueness). For $G \in C(\partial U)$ and $f \in C(U)$, there exists at most one solution $u \in C^2(U) \cap C(\overline{U})$ of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}.$$

Theorem 2.5 (6, Smoothness). Harmonic functions are smooth.

Theorem 2.6. For u harmonic,

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B_r(x_0))},$$

where

$$C_0 = \frac{1}{m(B_1)}, \quad C_k = \frac{(2^{n+1}nk)^k}{m(B_1)}$$

for every ball $B_r(x_0) \subset U$.

Theorem 2.7 (8, Liouville). For u bounded and harmonic,

$$Du = 0.$$

Theorem 2.8 (9, Representation formula). For $f \in C_c^2(\mathbb{R}^{n \geq 3})$, every bounded solution of Poisson's equation has form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) dy + C$$

for some constant C .

Theorem 2.9 (11, **Harnack's Inequality**). For a connected, bounded open set $V \subset U$,

$$\sup_V u \leq C_v \inf_V u$$

for every nonnegative harmonic function u in U .

Define **Green's function** by

$$G(x, y) = \Phi(y - r) - K_r(y),$$

where $K_x(y)$ is a Dirac kernel supported in $B_0(x)$.

Theorem 2.10. For $u \in C^2(\overline{U})$ a solution to Poisson's equation,

$$u(x) = - \int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x, y) dS(y) + \int_U f(y) G(x, y) dy,$$

where ν is the outward-facing unit normal vector on ∂U .

For $x \in \mathbb{R}_{\neq 0}^n$,

$$\tilde{x} = \frac{x}{|x|^2}$$

is the point **dual** to X with respect to $\partial B_1(\mathbf{0})$, and $x \mapsto \tilde{x}$ is the **inversion** through the unit sphere $\partial B_1(\mathbf{0})$.

Theorem 2.11 (15). For

$$u(x) = \frac{r^2 - |x|^2}{nm(B_1)r} \int_{\partial B_r(\mathbf{0})} \frac{g(y)}{|x-y|^n} dS(y)$$

with $g \in C(\partial B_r(\mathbf{0}))$,

$$u \in C^\infty(B_r^{\neq \mathbf{0}}(\mathbf{0})), \quad \Delta u = 0 \quad \text{in } B_1^{\neq \mathbf{0}}(\mathbf{0}), \quad \text{and} \quad \lim_{x \in B_r(\mathbf{0}) \rightarrow x_0} u(x) = g(x^0), \quad x^0 \in \partial B_r(\mathbf{0}).$$

Define the **energy functional**

$$I[w] = \int_U \frac{1}{2} |Dw|^2 - wf dx,$$

where w belongs to the **admissible set**

$$\mathcal{A} = \{w \in C^2(\overline{U}) \mid w = g \text{ on } \partial U\}.$$

2.3 Heat

The **heat equation** is defined by

$$u_t - \Delta u = 0$$

with nonhomogeneous variant

$$u_t - \Delta u = f.$$

The fundamental solution u has optimal special structure

$$u(x, t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

given by a heat equation invariant under the **dilation scaling**

$$u(x, t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t).$$

The fundamental solution to the heat equation is given by

$$\frac{1}{(4\pi t)^{\frac{n}{2}} e^{-\frac{|x|^2}{4t}}},$$

and the fundamental solution to the nonhomogeneous problem is given by

$$u(x, t) = \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) dy ds.$$

Define the **parabolic cylinder**

$$U_T = U \times (0, T]$$

so that its boundary Γ_T does not include the top.

Define a **heat ball** by

$$E(x, t; r) = \left\{ (y, s) \in \mathbb{R}^{n+1} \mid s \leq t, \quad \Phi(x - y, t - s) \geq \frac{1}{r^n} \right\}.$$

Theorem 2.12 (3). *For $u \in C_1^2(U_T)$ a solution to the heat equation,*

$$u(x, t) = \frac{1}{4r^n} \iint_{E(x, t; r)} u(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds$$

for $E(x, t; r) \subset U_T$.

Theorem 2.13 (4). *For $u \in C_1^2(U^T) \cap C(\overline{U}_T)$ a solution to the heat equation,*

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u,$$

and if U is connected,

$$u(x_0, t_0) = \max_{\overline{U}_T} u \implies Du = 0$$

for $t = t_0$.

Theorem 2.14 (9). *For every pair of natural numbers (k, l) , there exists a constant $C_{k, l}$ such that*

$$\max C\left(x, t; \frac{r}{2}\right) \left| D_x^k D_t^l u \right| \leq \frac{C_{k, l}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x, t; r))}$$

for $C(x, t; \frac{r}{2}) \subset C(x, t; r) \subset U_T$ and u a solution to the heat equation in U_T .

2.4 Wave

The **wave equation** is given by

$$u_{tt} - \Delta u = 0$$

with nonhomogenous variant

$$u_{tt} - \Delta = f.$$

For $n = 1$, the solution is given by **d'Alembert's formula**

$$u(x, t) = \frac{1}{2} [g(x+t) + g(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy.$$

For $n = 3$, the solution is given by **Kirchoff's formula**

$$4u(x, t) = \frac{1}{m(\partial B_t(x))} \int_{\partial B_t(x)} th(y) + g(y) + Dg(y) \cdot (y-x) dS(y).$$

For $n = 2$, taken as a special case of $n = 3$ (via **method of descent**), we have **Poisson's formula**

$$u(x, t) = \frac{1}{2} \cdot \frac{1}{m(B_t(x))} \int_{B_t(x)} \frac{tg(y) + t^2 h(y) + tDg(y) \cdot (y-x)}{\sqrt{(t^2 - |y-x|^2)}} dy.$$

Generally, for $n = 2k + 1$,

$$u(x, t) = \frac{1}{\gamma_n} \left[\left(\frac{\partial}{\partial t} \right) \left(\frac{1}{t} \cdot \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(\frac{t^{n-2}}{m(\partial B_t(x))} \int_{\partial B_t(x)} g ds \right) + \left(\frac{1}{t} \cdot \frac{d}{dt} \right)^{\frac{n-3}{2}} \left(\frac{t^{n-2}}{m(\partial B_t(x))} \int_{\partial B_t(x)} h dS \right) \right],$$

where

$$\gamma_n = \prod_{i=0}^k (2i+1).$$

Generally, for $n = 2k$,

$$u(x, t) = \frac{1}{\gamma_n} \left[\left(\frac{\partial}{\partial t} \right) \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(\frac{t^n}{m(B_t(x))} \int_{B_t(x)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} dy \right) + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(\frac{t^n}{m(B_t(x))} \int_{B_t(x)} \frac{h(y)}{\sqrt{t^2 - |y-x|^2}} dy \right) \right],$$

where

$$\gamma_n = \prod_{i=0}^k 2k.$$

That disturbances originating at some x propagate along a sharp wavefront in odd dimensions but continue to affect the field after the leading edge of the wavefront passes in even dimensions is **Huygens' principle**.

Theorem 2.15 (4). *For $n \geq 2$, a solution of the nonhomogeneous wave equation is given by*

$$u(x, t) = \int_0^t u(x, t; s) ds$$

where

$$\begin{cases} u_t(\cdot; s) - \Delta u(\cdot; s) = 0 & \text{in } \mathbb{R}^n \times (s, \infty) \\ u(\cdot; s) = 0, u_t(\cdot; s) = f(\cdot, s) & \text{on } \mathbb{R}^n \times \{t = s\}. \end{cases}$$

The integrant on the right of

$$u(x, t) = \frac{1}{4\pi} \int_{B_t(x)} \frac{f(y, t - |y - x|)}{|y - x|} dy,$$

a solution to the wave equation for $n = 3$, is a **retarded potential**.

Theorem 2.16 (5). *There exists at most one function $u \in C^2(\overline{U}_T)$ which solves*

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } U \times \{t = 0\}. \end{cases}$$

Define the **cone**

$$Cx_0, t_0 = \{(x, t) \mid t \in [0, t_0], d(x, x_0) \leq d(t_0, t)\}.$$

Theorem 2.17 (6). *For $u \equiv u_t \equiv 0$ on $B_{t_0}(x_0)$,*

$$u(C_{x+0, t+0}) \equiv 0.$$

3 Nonlinear First-Order PDEs

Use notation such that

$$F(Du, u, x) = 0 = F(p, z, x) = F(p_1, \dots, p_n, z, x_1, \dots, x_n),$$

and

$$\begin{aligned} D_p F &= (F_{p_1}, \dots, F_{p_n}) \\ D_z F &= F_z \\ D_x F &= (F_{x_1}, \dots, F_{x_n}). \end{aligned}$$

3.1 Complete Integrals and Envelopes

Use notation such that

$$(D_a u, D_{xa}^2 u) = \begin{bmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_1 a_n} & \cdots & u_{x_n a_n} \end{bmatrix}.$$

A $u(x; a) \in C^2(U \times A)$ is a **complete integral** in $U \times A$ if

$$u(x; a) \text{ solves the PDE } F(Du, u, x) = 0 \quad \forall a \in A$$

and

$$\dim(D_a u, D_{xa}^2 u) = n$$

everywhere.

For $u(x; a) \in C^1(U \times A)$, $D_a u(z; a) = 0$ solved for the parameter $a = \phi(x) \in C^1$ so $D_a u(x; \phi(x)) = 0$, the **envelope** of the functions $\{u(\cdot; a)\}_{a \in A}$ is

$$v(x) = u(x; \phi(x)).$$

The envelope v may be called the **singular integral** of the PDE $F(Du, u, x) = 0$.

Theorem 3.1 (1). For $u = u(\cdot; a)$ solves the PDE $F(Du, u, x)$ for all a ,

the envelope v solves the PDE also.

Use notation

$$a = (a_1, \dots, a_n) = (a', a_n).$$

The **general integral** for $C^1 \ni h' : A \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is the envelope $v'(x) \in C^1$ of the functions

$$u'(x; a') = u(x; a', h(a')).$$

3.2 Characteristics

The **characteristic equations** of the nonlinear first-order PDE $F(Du, u, x) = 0$ are given by

$$\begin{cases} \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)). \end{cases}$$

The **characteristics** are $\mathbf{p}(\cdot)$, $z(\cdot)$, and $\mathbf{x}(\cdot)$, and $(\cdot)$ is specifically the **projected characteristic** of the full characteristics $(\mathbf{p}(\cdot), z(\cdot), \mathbf{x}(\cdot)) \subset \mathbb{R}^{2n+1}$ onto $U \subset \mathbb{R}^n$.

Theorem 3.2 (1). For $u \in C^2(U)$ a solution to the nonlinear first-order PDE $F(Du, u, x) = 0$ in U , $\mathbf{x}(\cdot)$ a solution to the ODE $\dot{\mathbf{x}}(s)$,

$$\mathbf{p}(\cdot) \text{ solves the ODE } \dot{\mathbf{p}}(s) \text{ and } z(\cdot) \text{ solves the ODE } \dot{z}(s)$$

where $\mathbf{x}(s) \in U$.

Define the **compatibility conditions**

$$p_i(0) = g_{x_i}(x_0), \quad i < n, \quad \text{and} \quad F(p(0), z(0), x(0)) = 0,$$

where $z(0) = g(\mathbf{x}(0))$ makes the compatibility conditions **admissible**.

Theorem 3.3 (2, Local Existence Theorem). For $u(x) = z(\mathbf{y}(x), s(x))$ and $\mathbf{p}(x) = \mathbf{p}(\mathbf{y}(x), s(x))$, where $x = \mathbf{x}(y, s)$ and $y = \mathbf{y}(x)$, and $s = s(x)$,

$$u \text{ solves the PDE } F(Du(x), u(x), x) = 0 \text{ with boundary condition } u(x) = g(x).$$

Define the **Lagrangian** $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ a smooth function with

$$L = L(q, x) = L(q_n, \dots, q_1, x_1, \dots, x_n)$$

and

$$\begin{cases} D_q L = (L_{q_1} \cdots L_{q_n}) \\ D_x L = (L_{x_1} \cdots L_{x_n}) \end{cases}$$

Define the **action functional** by

$$I[\mathbf{w}(\cdot)] = \int_0^t L(\mathbf{w}(s), \mathbf{w}(s)) ds$$

for functions of the **admissible class**

$$\mathcal{A} = \{\mathbf{w}(\cdot) \in C^2([0, t]; \mathbb{R}^n) \mid \mathbf{w}(0) = y, \quad (t) = x\}.$$

Theorem 3.4 (1). *For the system of Euler-Lagrange equations*

$$-\frac{d}{ds} (D_q L(\dot{\mathbf{x}}(s), \mathbf{x}(s)) + D_x L(\dot{\mathbf{x}}(s), \mathbf{x}(s))) = 0,$$

$\mathbf{x}(s)$ is a solution.

The **Hamiltonian** H for a Lagrangian L is

$$H(p, x) = p \cdot \mathbf{q}(p, x) - L(\mathbf{q}(p, x), x),$$

where $\mathbf{q}(\cdot, \cdot)$ is defined implicitly by the hypothesis that $p = D_q L(q, x)$ is uniquely solvable for q as a smooth function $q = \mathbf{q}(p, x)$.

Theorem 3.5 (2). *For Hamilton's equations*

$$\begin{aligned} \dot{\mathbf{x}}(s) &= D_p H(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{\mathbf{p}}(s) &= -D_x H(\mathbf{p}(s), \mathbf{x}(s)) \end{aligned}$$

and solutions $\mathbf{x}(\cdot)$ and $\mathbf{p}(\cdot)$, and the mapping

$$D_s H(\mathbf{p}(s), \mathbf{x}(s)) = \mathbf{0}.$$

Suppose the Lagrangian L is now convex with $\lim_{q \rightarrow \infty} \frac{L(q)}{|q|} = \infty$. The **Legendre transform** of L is given by

$$H = L^*(p) = \sup_q \{p \cdot q - L(q)\}.$$

Theorem 3.6 (4, Hopf-Lax formula). *The solution of the minimization problem*

$$u(x, t) = \inf \left\{ \int_0^t L(\dot{\mathbf{w}}(s)) ds + g(y) \mid \mathbf{w}(0) = y, \mathbf{w}(t) = x \right\}$$

is given by

$$u(x, t) = \min_y \left\{ tL\left(\frac{x - y}{t}\right) + g(y) \right\}.$$

Theorem 3.7 (5). *For u given by the Hopf-Lax formula differentiable at (x, t) ,*

$$u_t(x, t) + H(Du(x, t)) = 0.$$

A function $H \in C^2$ is **uniformly convex** if

$$\sum_{i,j=1}^n H_{p_i, p_j}(p) \xi_i \xi_j \geq \theta |\xi|^2$$

for some $\theta \in \mathbb{R}^+$ for all $p, \xi \in \mathbb{R}^n$.

A Lipschitz continuous function u is a weak solution of the initial-value problem if

$$u(x + z, t) - u(x, t) + u(x - z, t) \leq C \left(1 + \frac{1}{t}\right) |z|^2$$

for some constant $C \geq 0$, $u(x, 0) = g(x)$, and $u_t(x, t) + H(Du(x, t)) = 0$ a.e. (x, t) .

Theorem 3.8 (7). *For $H \in C^2$ and g Lipchitz continuous, there exists at most one solution to the initial-value problem.*

A $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an **integral solution** of

$$\begin{cases} u_t + F(u)_x & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

if

$$\int_0^\infty \int_{-\infty}^\infty uv_t F(u) v_x \, dx dt + \int_{-\infty}^\infty gv \, dx|_{t=0} = 0$$

for every test function $v \in \mathcal{D}(U)$.

The **Rankine–Hugoniot condition** refers to

$$F(u_l) - F(u_r) = \dot{s}(u_l - u_r),$$

where the subscripts l and r refer to the respective limits, and $\dot{s}$ the speed of the curve C .

A curve of discontinuity for u is a **shock** if both the Rankine–Hugoniot identity and **entropy condition**,

$$F'(u_l) > \dot{s} > F'(u_r),$$

hold.

Let

$$h(x) = \int_0^x g(y) \, dy.$$

Theorem 3.9 (1, Lax–Oleinik formula). *For $F : \mathbb{R} \rightarrow \mathbb{R}$ smooth, uniformly convex, and $G \in L^\infty(\mathbb{R})$, there exists for all but at most countably many $x \in \mathbb{R}$ a unique point $y(x, t)$ with*

$$\min y \in \mathbb{R} \left\{ tL \left(\frac{x - y}{t} \right) + h(y) \right\} = tL \left(\frac{x - y(x, t)}{t} \right) + h(y(x, t))$$

and

$$u(x, t) = \frac{\partial}{\partial x} \left[\min y \in \mathbb{R} \left\{ tL \left(\frac{x - y}{t} \right) + h(y) \right\} \right] = \left((F')^{-1} \left(\frac{x - y(x, t)}{t} \right) \right)$$

is an integral solution of the initial-value problem.

A function $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an **entropy solution** of the initial value problem if it is an integral solution and

$$u(x + z, t) - u(x, t) \leq C \left(1 + \frac{1}{t} \right) z$$

for some constant $C \geq 0$ a.e. $x, z \in \mathbb{R}$, $z > 0$.

Theorem 3.10 (3). *For F convex and smooth, there exists at most one entropy solution to the initial-value problem.*

The initial-value problem with piecewise-constant initial function

$$g(x) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0 \end{cases}$$

is **Riemann's problem** for the scalar conservation law, where $u_l, u_r \in \mathbb{R}$ are distinct **initial states**.

Theorem 3.11 (4). For $u_l > u_r$, the unique entropy solution of the Riemann problem is

$$u(x, t) = \begin{cases} u_l & \text{if } \frac{x}{t} < \eta, \\ u_r & \text{if } \frac{x}{t} > \eta, \end{cases}$$

where

$$\eta = \frac{F(u_l) - F(u_r)}{u_l - u_r},$$

otherwise, the unique entropy solution is

$$u(x, t) = \begin{cases} u_l & \text{if } \frac{x}{t} < F'(u_l), \\ G(\frac{x}{t}) & \text{if } F'(u_l) < \frac{x}{t} < F'(u_r), \\ u_r & \text{if } \frac{x}{t} > F'(u_r). \end{cases}$$

Theorem 3.12 (5). For an entropy solution u to the initial-value problem posed, there exists $C > 0$ such that

$$|u(x, t)| \leq \frac{C}{\sqrt{t}}$$

for all $x \in \mathbb{R}$, $t > 0$.

Define an **N-wave** with constants $p, q \geq 0$, $d > 0$, and velocity σ by the function

$$N(x, t) = \begin{cases} \frac{1}{d}(\frac{x}{t} - \sigma) & \text{if } -\sqrt{pd}t < x < \sigma t < \sqrt{qd}t, \\ 0 & \text{else.} \end{cases}$$

Theorem 3.13 (6). There exists C such that

$$\int_{-\infty}^{\infty} |u(\cdot, t) - N(\cdot, t)| dx \leq \frac{C}{\sqrt{t}}$$

4 Representing Solutions

4.1 Similarity Solutions

A solution u to a PDE of the form

$$u(x, t) = v(x - \sigma t)$$

is a **traveling wave** with speed σ and **profile** v , and in general

$$u(x, t) = v(y \cdot x - \sigma t)$$

is a **plane wave** with **wavefront** normal to $y \in \mathbb{R}^n$ and velocity $\frac{\sigma}{|y|}$.

A solution u to a PDE of the form

$$u(x, t) = c^{i(y \cdot x + \omega t)}$$

has **frequency** ω and **wave numbers** $\{y_i\}_{i=1}^n$.

Define **Airy's equation** by

$$u_t + u_{xxx} = 0.$$

The quality of waves of different frequencies propagating at different velocities is **dispersion**.

Schrödinger's equation is given by

$$iu_t + \Delta u = 0.$$

The **Kortewg–de Vries (KdV)** equation is given by

$$u_t + 6uu_x + u_{xxx} = 0.$$

A solution u to a PDE of form $u(x, t) = v(x - \sigma t)$ with

$$v, v', v'' \rightarrow 0$$

as $|s| \rightarrow \infty$ is a **solitary wave**, and a solution of form

$$u(x, t) = \frac{\sigma}{2} \operatorname{sech}^2 \left(\frac{\sqrt{\sigma}}{2} (x - \sigma t - c) \right)$$

is a **soliton**.

In a dynamical system, a **heteroclinic orbit** is a solution to PDEs which starts at one equilibrium point and ends at another.

The **porous medium equation** is given by

$$u_t - \Delta(u^\gamma) = 0$$

with $u \geq 0$, $\gamma > 1$.

4.2 Transform Solutions

The **Laplace transform** of $u \in L^1(\mathbb{R}_+)$ is given by

$$u^\#(s) = \int_0^\infty e^{-st} u(t) dt$$

for $s \geq 0$.

4.3 Nonlinear to Linear

The **Hopf–Cole transformation** is given by

$$w = \exp \left\{ \frac{-bu}{a} \right\},$$

where $w = \phi(u)$ is a smooth function.

The **hodograph transform** refers to the process of converting particular quasi-linear systems of PDE into linear systems by reversing the roles of dependent and independent variables.

The **Legendre transform** refers to the hodograph transform with the inclusion of the gradient of a solution as new, independent variables.

4.4 Asymptotics

A **singular perturbation** is the modification of a PDE by adding the product of some error ϵ and a higher-order term.

Homogenization theory studies the effects of high-frequency oscillations in coefficients upon the solutions of PDEs.

For the equation

$$\frac{\partial^k u}{\partial x_n^k} - \frac{1}{a_{(0,\dots,0,k)}} \left[\sum_{\substack{\alpha \neq (0,\dots,0,k) \\ |\alpha|=k}} a_\alpha D^\alpha u + a_0 \right],$$

the plane $\Gamma = \{x_n = 0\}$ is **noncharacteristic** for

$$\sum_{|\alpha|=k} a_\alpha (D^{k-1}u, \dots, u, x) D^\alpha u + a_0 (D^{k-1}u, \dots, u, x) = 0$$

if $a_{(0,\dots,k)}$ is nonzero for all values of its arguments.

Theorem 4.1 (1). *For Γ noncharacteristic of the PDE as above and u a smooth solution satisfying the Cauchy conditions, all partial derivatives of u along Γ in terms of Γ are computable, as well as functions $g_0, \dots, g_{k-1}$ and coefficients $a_\alpha (|\alpha| = k)$, a_0 .*

5 Sobolev

The **Sobolev space**

$$W^{k,p}(U)$$

is every locally summable scalar function u such that for each multi-index α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Define the α^{th} **weak partial derivative** of $u \in L^1_{\text{loc}}(U)$ to be the v such that

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v \phi \, dx$$

for all test functions $\phi \in \mathcal{D}(U)$.

Let

$$W_0^{k,p}(U)$$

be the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Theorem 5.1 (Trace). *For U bounded and $\partial U \in C^1$, there exists a bounded $T \in \mathcal{L}(W^{1,p}(U), L^p(\partial U))$ with*

$$Tu = u|_{\partial U}, \quad u \in W^{1,p}(U) \cap C(\bar{U})$$

and

$$\|Tu\|_{L^p(\partial U)} \leq K_{p,U} \|u\|_{W^{1,p}(U)}, \quad u \in W^{1,p}(U)$$

for some constant $C_{p,U}$. Tu is then the **trace** of u on ∂U .

Theorem 5.2 (2). *For U bounded, $\partial U \in C^1$, and $u \in W^{1,p}(U)$,*

$$u \in W_0^{1,p}(U) \iff Tu = 0 \text{ on } \partial U.$$

5.1 Inequalities

For $1 \leq p < n$, the **Sobolev conjugate** of p is

$$p^\star = \frac{np}{n-p},$$

i.e.,

$$\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{n}$$

for $p^\star > p$.

Theorem 5.3 (Gagliardo–Nirenberg–Sobolev inequality). *For $1 \leq p \leq n$ and $u \in C_c^1(\mathbb{R}^n)$, there exists a constant $C_{p,n}$ with*

$$\|u\|_{L^{p^\star}(\mathbb{R}^n)} \leq C_{p,n} \|Du\|_{L^p(\mathbb{R}^n)}.$$

Theorem 5.4 (2). *For $u \subset \mathbb{R}^n$ open and bounded, $\partial U \in C^1$, $1 \leq p < n$, $u \in W^{1,p}(U)$, we have that $u \in L^{p^\star}(U)$ with*

$$\|u\|_{L^{p^\star}(U)} \leq C_{p,n,U} \|u\|_{W^{1,p}(U)}$$

for some constant $C_{p,n,U}$.

Theorem 5.5 (3). *For $U \subset \mathbb{R}^n$ bounded and open and $u \in W_0^{1,p}(U)$ with $1 \leq p < n$,*

$$\|u\|_{L^q(U)} \leq C_{p,q,n,U} \|Du\|_{L^p(U)}$$

for every $q \in [1, p^\star]$ for some constant $C_{p,q,n,U}$.

Theorem 5.6 (4, Morrey's inequality). *For $n < p \leq \infty$,*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq K_{p,n} \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

with some $K_{p,n}$ for every $u \in C^1(\mathbb{R}^n)$ and

$$\gamma = 1 - \frac{n}{p}.$$

$u^\star$ is the **version** of a given function u if $u = u^\star$ a.e.

Theorem 5.7 (5). *For $U \subset \mathbb{R}^n$ bounded and open, $\partial U \in C^1$, $n < p \leq \infty$, $u \in W^{1,p}(U)$, and u having version $u^\star \in C^{0,\gamma}(\overline{U})$ with $\gamma = 1 - \frac{n}{p}$,*

$$\|u^\star\|_{C^{0,\gamma}(\overline{U})} \leq C_{p,n,U} \|u\|_{W^{1,p}(U)}$$

for some constant $C_{p,n,U}$.

Theorem 5.8 (1, Poincaré's inequality). *For $U \subset \mathbb{R}^n$ bounded, connected, and open, $\partial U \in C^1$, $1 \leq p \leq \infty$,*

$$\left\| u - \left[\frac{1}{m(u)} \int_U u \right] \right\|_{L^p(U)} \leq C_{n,p,U} \|Du\|_{L^p(U)}$$

for every function $u \in W^{1,p}(U)$ for some constant $C_{n,p,U}$.

Theorem 5.9 (4). *For $U \subset \mathbb{R}^n$ open and bounded and $\partial U \in C^1$,*

$$u \text{ is Lipschitz continuous} \iff u \in W^{1,\infty}(U).$$

Define the space $H^k(\mathbb{R}^n)$ to be every function $u \in L^2(\mathbb{R}^n)$ with

$$(1 + |y|^k) \hat{u} \in L^2(\mathbb{R}^n).$$

6 Second-Order Elliptic Equations

Consider the initial-value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

and L a second-order partial differential operator having form

$$Lu = - \sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u.$$

The PDE L is **uniformly elliptic** if there exists $\theta > 0$ with

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$. One may characterize ellipticity by the matrix a^{ij} positive definite with smallest eigenvalue at least θ .

The **bilinear form** $B[\cdot, \cdot]$ associated with the divergence form elliptic operator L is given by

$$B[u, v] = \int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v + uv \, dx$$

for $u, v \in H^1_0(U)$.

A **weak solution** of the boundary-value problem above $u \in H^1_0$ is defined by

$$B[u, v] = \langle f, v \rangle$$

for all $V \in H^1_0(U)$.

Theorem 6.1 (2). *There exist $\alpha, \beta > 0$ and $\gamma \geq 0$ with*

$$|B[u, v]| \leq \alpha \|u\|_{H^1_0(U)} \|v\|_{H^1_0(U)}$$

and

$$\beta \|u\|_{H^1_0(U)}^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2$$

for all $u, v \in H^1_0(U)$.

Theorem 6.2 (3). *There exists $\gamma \geq 0$ such that for $u \geq \gamma$ and $f \in L^2(U)$, there exists a unique weak solution $u \in H^1_0(U)$ of the boundary-value problem posed above.*

The **adjoint** of L , L^* , is given by

$$L^*v = - \sum_{i,j=1}^n (a^{ij}v_{x_j})_{x_i} - \sum_{i=1}^n b^i v_{x_i} + (c - \sum_{i=1}^n b_{i,x_i})v.$$

Theorem 6.3 (4, Fredholm alternative). *For every $f \in L^2(U)$, there exists a unique weak solution u of the boundary-value problem above if and only if there does not exist a weak solution $u \neq 0$ of the corresponding homogenous problem.*

Theorem 6.4 (5). *There exists an at most nondecreasing countable set $\Sigma \subset \mathbb{R}$ such that the boundary-value problem*

$$\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

has a unique weak solution for each $f \in L^2(U) \iff \lambda \notin \Sigma$.

Theorem 6.5 (3, Strong maximum principle). *For $U \subset \mathbb{R}^n$ connected, open, and bounded, $u \in C^2(U) \cap C(\overline{U})$, and $u \equiv 0$ in U ,*

$$Lu (\leq \vee \geq) 0 \text{ in } U \text{ and } \max_{\overline{U}} u = \max_U u \implies Du = 0 \text{ in } U.$$

Theorem 6.6 (3, Principal eigenvalue for nonsymmetric elliptic operators). *There exists a real eigenvalue λ_1 for L with zero boundary conditions and $\operatorname{Re}(\lambda) \leq \lambda_1$ for any other eigenvalue λ . The corresponding eigenfunction w_1 is positive within U , and λ_1 is **simple**, meaning any solution u of the boundary problem posed is a multiple of w_1 .*

7 Linear Evolution Equations

Consider the parabolic initial/boundary-value problem

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g & \text{on } U \times \{t = 0\}, \end{cases}$$

for L a second-order PDO for each time t .

A PDO

$$\frac{\partial}{\partial t} + L$$

is **uniformly parabolic** if there exists $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x, t) \xi_i \xi_j \geq \theta |\xi|^2$$

for all $(x, t) \in U_T$, $\xi \in \mathbb{R}^n$.

A function

$$\mathbf{u} \in L^2(0, T; H_0^1(U)), \quad \mathbf{u}' \in L^2(0, T; H^{-1}(U))$$

is a **weak solution** of the parabolic boundary-value problem posed if

$$\langle \mathbf{u}', v \rangle + B[\mathbf{u}, v; t] = \langle \mathbf{f}, v \rangle$$

for every $v \in H_0^1(U)$ and a.e. $0 \leq t \leq T$, and

$$\mathbf{u}(0) = g.$$

Theorem 7.1 (2). *There exists a constant $C_{U,T,L}$ such that*

$$\max_{0 \leq t \leq T} \|\mathbf{u}_m(t)\|_{L^2(U)} + \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))} \leq C_{U,T,L} (\|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)})$$

for $m \in \mathbb{Z}_+$.

Theorem 7.2 (3,4). *There exists a unique weak solution to the posed initial/boundary-value problem.*

The **essential supremum** of f , denoted

$$\operatorname{ess} f$$

is defined as the supremum of f such that the measure of $f^{-1}(X) = \operatorname{ess} f$ has $m(X) > 0$.

Theorem 7.3 (6, Higher regularity). *For $g \in H^{2m+1}(U)$, $\frac{d^k \mathbf{f}}{dt^k} \in L^2(0, T; H^{2m-2k}(U))$, $k \in [m]$, and*

$$\begin{cases} g_0 = g \in H_0^1(U), \\ g_1 = \mathbf{f}(0) - Lg_0 \in H_0^1(U), \\ \dots, \\ g_m = \frac{d^{m-1} \mathbf{f}}{dt^{m-1}}(0) - Lg_{m-1} \in H_0^1(U) \end{cases}$$

we have

$$\frac{d^k \mathbf{u}}{dt^k} \in L^2(0, T; H^{2m+2-k}(U)), \quad k \in [m+1]$$

and

$$\sum_{k=0}^{m+1} \left\| \frac{d^k \mathbf{u}}{dt^k} \right\|_{L^2(0, T; H^{2m+2-k}(U))} \leq C_{m,U,T,L} \left(\sum_{k=0}^m \left\| \frac{d^k \mathbf{f}}{dt^k} \right\|_{L^2(0, T; H^{2m-2k}(U))} + \|g\|_{H^{2m+1}(U)} \right)$$

for some constant $C_{m,U,T,L}$.

7.1 Hyperbolic

Define the initial/boundary-value problem

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g, \quad u_t = h & \text{on } U \times \{t = 0\}, \end{cases}$$

where L is a second-order partial differential operator at each time t .

A PDO

$$\frac{\partial^2}{\partial^2 t} + L$$

is **uniformly hyperbolic** if there exists $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x, t) \xi_i \xi_j \geq 0 |\xi|^2$$

for all $(x, t) \in U_T$, $\xi \in \mathbb{R}^n$.

A function

$$\mathbf{u} \in L^2(0, T; H_0^1(U)), \quad \mathbf{u}' \in L^2(0, T; L^2(U)), \quad \mathbf{u}'' \in L^2(0, T; H^{-1}(U))$$

is a **weak solution** of the hyperbolic initial/boundary-value problem posed if

$$\langle \mathbf{u}'', v \rangle + B[\mathbf{u}, v; t] = \langle \mathbf{f}, v \rangle$$

for each $v \in H_0^1(U)$ and a.e. $0 \leq t \leq T$ and

$$\mathbf{u}(0) = g, \quad \mathbf{u}'(0) = h.$$

Theorem 7.4 (3,4). *there exists a unique weak solution to the hyperbolic initial/boundary-value problem posed.*

Consider systems of linear first-order partial differential equations of the form

$$\mathbf{u}_t + \sum_{j=1}^n \mathbf{B}_j \mathbf{u}_{x_j} = \mathbf{f} \quad (\text{in } \mathbb{R}^n \times (0, \infty)), \quad \mathbf{u} = \mathbf{g} \quad (\text{on } \mathbb{R}^n \times \{t = 0\}).$$

The system of PDEs above is **hyperbolic** if $\mathbf{B}(x, t; y) \in \mathbb{R}^{m \times m}$ is always diagonalizable.

A function

$$\mathbf{u} \in L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m)), \quad \mathbf{u}' \in L^2(0, T; L^2(\mathbb{R}^n; \mathbb{R}^m))$$

is a **weak solution** of the initial-value problem above for the symmetric hyperbolic system if

$$\langle \mathbf{u}', \mathbf{v} \rangle + B[\mathbf{u}, \mathbf{v}; t] = \langle \mathbf{f}, \mathbf{v} \rangle$$

for each $\mathbf{v} \in H^1(\mathbb{R}^n; \mathbb{R}^m)$ and a.e. $0 \leq t \leq T$ and

$$\mathbf{u}(0) = \mathbf{g}.$$

Theorem 7.5 (1). *For every $\epsilon > 0$, there exists a unique solution $\mathbf{u}^\epsilon$ of*

$$\begin{cases} \mathbf{u}_t^\epsilon - \epsilon \Delta \mathbf{u}^\epsilon + \sum_{j=1}^n \mathbf{B}_j \mathbf{u}_{x_j}^\epsilon = \mathbf{f} & \text{in } \mathbb{R}^n \times (0, T], \\ \mathbf{u}^\epsilon = \mathbf{g}^\epsilon & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

with

$$\mathbf{u}^\epsilon \in L^2(0, T; H^3(\mathbb{R}^n; \mathbb{R}^m)), \quad \mathbf{u}^{\epsilon'} \in L^2(0, T; H^1(\mathbb{R}^n; \mathbb{R}^m)).$$

Theorem 7.6 (3,4). *There exists a unique weak solution to the posed system of PDEs.*

A family of bounded linear operators over X , $\{S(t)\}_{t \geq 0}$ with

$$S(0)u = u \quad \text{and} \quad t \mapsto S(t)u \text{ continuous into } X$$

is a **semigroup**.

A **contraction semigroup** is a semigroup with

$$\|S(t)\| \leq 1.$$

Define the **infinitesimal generator** of the semigroup $\{S(t)\}_{t \geq 0}$ $A : D(A) \rightarrow X$ by

$$D(A) = \left\{ u \in X \mid \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t} \in X \right\}$$

and

$$Au = \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t} \quad (u \in D(A)),$$

making $D(A)$ the *domain* of A .

Theorem 7.7 (2). *The domain $D(A)$ is dense in X , and A is a closed operator.*

A $\lambda \in \mathbb{R}$ belongs to $\rho(A)$, the **resolvent set** of A , if

$$\lambda I - A : D(A) \rightarrow X$$

is invertible. The **resolvent operator** is then defined by

$$R_\lambda u = (\lambda I - A)^{-1}u$$

Theorem 7.8 (4, Hille–Yosida). *For $A \in \mathcal{L}(X)$ closed and densely-defined, A is the generator of a contraction semigroup $\{S(t)\}_{t \geq 0}$ iff*

$$(0, \infty) \subset \rho(A) \quad \text{and} \quad \|R_\lambda\| \leq \frac{1}{\lambda}$$

for $\lambda > 0$.

Theorem 7.9 (5,6). *The operator A generates a γ -contraction semigroup $\{S(t)\}_{t \geq 0}$ on $L^2(U)$ and on $H_0^1(U) \times L^2(U)$.*

8 Calculus of Variations

Define the class of **variational problems** as PDE of the form

$$A[u] = 0,$$

where $A[\cdot]$ is the “derivative” of an “energy” functional.

A Lagrangian L is **null** if the system of Euler-Lagrange equations

$$-\sum_{i=1}^n \left(L_{p_i^k}(D\mathbf{u}, \mathbf{u}, x) \right)_{x_i} + L_z YK(D\mathbf{u}, \mathbf{u}, x) = 0 \quad \text{in } U, \quad k \in [m],$$

is automatically solved by all smooth functions $\mathbf{u}$.

Theorem 8.1 (1). *For L a null Lagrangian and $\mathbf{u}, \tilde{\mathbf{u}} \in C^2(\overline{U}, \mathbb{R}^m)$ with*

$$\mathbf{u} \equiv \tilde{\mathbf{u}} \quad \text{on } \partial U,$$

$$I[\mathbf{u}] = I[\tilde{\mathbf{u}}].$$

Define the **cofactor matrix** of $A \in \mathbb{R}^{n \times n}$, $\text{cof } A$, by the matrix where $(k, i)^{\text{th}}$ entry has $(\text{cof } A)_i^k = (-1)^{i+k} d(A)_i^k$ for

$$d(A)_i^k = \det M,$$

where M is the $(n-1) \times (n-1)$ matrix obtained by deleting the k^{th} -row and i^{th} -column from A .

Theorem 8.2 (2).

$$L(P) = \det P \quad (P \in \mathbb{R}^{n \times n})$$

is a null Lagrangian.

Theorem 8.3 (3, Brouwer’s Fixed Point). *For $\mathbf{u} : B_1(0) \rightarrow B_1(0)$ continuous, there exists $x \in B_1(0)$ with*

$$\mathbf{u}(x) = x.$$

For $1 < q < \infty$, constants $\alpha > 0, \beta \geq 0$ with

$$L(p, z, x) \geq \alpha|p|^q - \beta$$

for all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $x \in U$, we have

$$I[w] \geq \alpha \|Dw\|_{L^q(U)}^q - \gamma$$

for $\gamma = \beta|U|$, the **coercivity condition** on $I[\cdot]$. The class of admissible functions w is thence

$$\mathcal{A} = \{w \in W^{1,q}(U) \mid w = g \text{ on } \partial U \text{ by trace}\}.$$

For

$$m = \inf_{w \in \mathcal{A}} I[w] \quad I[u_k] \rightarrow m \quad \text{as } k \rightarrow \infty,$$

$\{u_k\}_{k=1}^\infty$ is a **minimizing sequence**.

A function $I[\cdot]$ is **sequentially weakly lower semi-continuous** on $W^{1,q}(U)$ if

$$u_k \rightarrow u \text{ weakly in } W^{1,q}(U) \implies I[u] \leq \liminf_{k \rightarrow \infty} I[u_k].$$

Theorem 8.4 (1). *For L bounded from below and $p \mapsto L(p, z, x)$ convex for each $x \in \mathbb{R}$, and $x \in U$,*

$$I[\cdot] \text{ is weakly lower semi-continuous on } W^{1,q}(U).$$

Theorem 8.5 (2). *For L satisfying the coercivity condition and convex in p with nonempty admissible set $\mathcal{A}$, there exists $u \in \mathcal{A}$ with*

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

A $u \in \mathcal{A}$ is a **weak solution** of the boundary-value problem

$$\begin{cases} -\sum_{i=1}^n (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0 & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases}$$

if

$$\int_U \sum_{i=1}^n L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v \, dx = 0$$

for all $v \in W_0^{1,q}(U)$.

A Lagrangian

$$L(P, z, x) = F(P, \det P, z, x)$$

is **polyconvex** if

$$(P, r) \mapsto F(P, r, z, x) \text{ is convex for each fixed } z \in \mathbb{R}^m, x \in \mathbb{R}^n.$$

Theorem 8.6 (8). *For $n < q < \infty$ and L bounded below and polyconvex,*

$$I[\cdot] \text{ is weakly lower semi-continuous on } W^{1,q}(U; \mathbb{R}^n).$$

Theorem 8.7 (2, Lagrange multiplier). *For $u \in \mathcal{A}$ with $I[u] = \min_{w \in \mathcal{A}} I[w]$, there exists $\lambda \in \mathbb{R}$ with*

$$\int_U Du \cdot Dv \, dx = \lambda \int_U g(u) v \, dx$$

for all $v \in H_0^1(U)$.

Theorem 8.8 (4). For $u \in \mathcal{A}$ the unique solution to $I[u] = \min_{w \in \mathcal{A}} I[w]$,

$$\int_U Du \cdot D(w - u) \, dx \geq \int_U f(w - u) \, dx$$

for all $w \in \mathcal{A}$; this is a **variational inequality**.

The restriction

$$\begin{cases} u \geq h, -\Delta u \geq f & \text{a.e. in } U, \\ -\Delta u = f & \text{on } U \cap \{u > h\}, \end{cases}$$

is the **free boundary**.

Consider now the energy minimization problem over

$$I[\mathbf{w}] = \frac{1}{2} \int_U |D\mathbf{w}|^2 \, dx$$

for

$$\mathcal{A} = \{\mathbf{w} \in H^1(U; \mathbb{R}^m) \mid \mathbf{w} = \mathbf{g} \text{ on } \partial U, |\mathbf{w}| = 1 \text{ a.e.}\},$$

a **harmonic map**.

A functional $I \in C^1(H; \mathbb{R})$ satisfies the **Palais–Smale compactness condition** if every sequence $\{u_k\}_{k=1}^\infty \subset H$ has functional sequence $\{I[u_k]\}_{k=1}^\infty$ bounded and $I'[u_k] \rightarrow 0$ in H is pre-compact in H .

Theorem 8.9 (2, Mountain Pass). For $I \in \mathcal{C}$ satisfying the Palais–Smale condition, $I[0] = 0$, the existence of constants $r, a > 0$ with $I[u] \geq a$ when $\|u\| = r$, and an element $v \in H$ with $\|v\| > r$ and $I[v] \leq 0$,

$$c = \inf_{\mathbf{g} \in \Gamma} \max_{0 \leq t \leq 1} I[\mathbf{g}(t)]$$

is a critical value of I , where

$$\Gamma = \{\mathbf{g} \in C([0, 1]; H) \mid \mathbf{g}(0) = 0, \mathbf{g}(1) = v\}.$$

9 Nonvariational Methods

A vector field $\mathbf{a}$ with

$$(\mathbf{a}(p) - \mathbf{a}(q)) \cdot (p - q) \geq 0$$

for all $p, q \in \mathbb{R}^n$ is **monotone**.

Theorem 9.1 (1, Banach’s Fixed Point). For a nonlinear mapping A a **strict contraction**, i.e.,

$$\|A[u] - A[\tilde{u}]\| \leq \gamma \|u - \tilde{u}\|$$

for some $\gamma < 1$,

A has a unique fixed point.

$\tilde{u} \in H^1(U)$ is a **weak supersolution** of the nonlinear Poisson equation if

$$\int_U D\tilde{u} \cdot Dv \, dx \geq \int_U f(\tilde{u})v \, dx$$

for each $v \in H_0^1(U)$, $v \geq 0$ a.e.

An open $U \subset \mathbb{R}^n$ is **star-shaped** with respect to 0 if $\{\lambda x \mid 0 \leq \lambda \leq 1\}$ lies in $\tilde{U}$ for each $x \in \tilde{U}$.
A function $I : \mathcal{H} \rightarrow (-\infty, \infty]$ is **lower semi-continuous** if

$$u_k \rightarrow u \implies I[u] \leq \liminf_{k \rightarrow \infty} I[u_k], \quad (u \in \mathcal{H}).$$

Such an I is **proper** if its image does not include ∞ .

For I convex and proper,

$$\partial I[u] = \{v \in \mathcal{H} \mid I[w] \geq I[u] + (v, w - u) \ \forall w \in \mathcal{H}\}.$$

The mapping $\partial I : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is the **subdifferential** of I .

For $\lambda > 0$ the **nonlinear resolvent** $J_\lambda : \mathcal{H} \rightarrow D(\partial I)$ is defined by

$$J_\lambda[w] = u,$$

where u the unique solution of $w \in u + \lambda \partial I[u]$.

For $\lambda > 0$ the **Yosida approximation** $A_\lambda : \mathcal{H} \rightarrow \mathcal{H}$ is defined by

$$A_\lambda[w] = \frac{w - J_\lambda[w]}{\lambda}$$

for all $w \in \mathcal{H}$.

Theorem 9.2 (Solution to gradient flow). *For $u \in D(\partial I)$ there exists a unique*

$$\mathbf{u} \in C([0, \infty); H), \quad \mathbf{u}' \in L^\infty(0, \infty; H)$$

with

$$\begin{aligned} \mathbf{u}(0) &= u, \\ \mathbf{u}(t) &\in D(\partial I) \quad \text{for } t > 0, \text{ and} \\ \mathbf{u}'(t) &\in -\partial I[\mathbf{u}(t)] \quad \text{for a.e. } t \geq 0. \end{aligned}$$

Theorem 9.3 (4).

$$D(\partial I) = H^2(U) \cap H_0^1(U).$$

10 Hamilton–Jacobi Equations

The **Hamilton–Jacobi equations** are given by

$$\begin{cases} u_t + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

The **vanishing viscosity** variant is given by

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon, x) - \epsilon \Delta u^\epsilon = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u^\epsilon = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

A bounded and uniformly continuous function u is a **viscosity solution** of the above initial-value problem for the Hamilton–Jacobi equation if

$$\begin{aligned} u &= g && \text{on } \mathbb{R}^n \times \{t = 0\}, \\ v_t(x_0, t_0) + H(Dv(x_0, t_0), x_-) &\leq 0 && \text{if } u - v \text{ has local max at } (x_0, t_0), \text{ and} \\ v_t(x_0, t_-) + H(Dv(x_0, t_0), x_-) &\geq 0 && \text{if } u - v \text{ has local min } (x_0, t_0). \end{aligned}$$

11 Systems of Conversation Laws

A $\mathbf{u} \in L^\infty(\mathbb{R}^n \times (0, \infty); \mathbb{R}^m)$ is an **integral solution** of the initial-value problem

$$\begin{cases} \mathbf{u}_t + \mathbf{F}(\mathbf{u})_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ \mathbf{u} = \mathbf{g} & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

if

$$\int_0^\infty \int_{-\infty}^\infty \mathbf{u} \cdot \mathbf{v}_t + \mathbf{F}(\mathbf{u}) \cdot \mathbf{v}_x \, dx dt + \int_{-\infty}^\infty \mathbf{g} \cdot \mathbf{v} \, dx|_{t=0} = 0$$

for all $\mathbf{v} \in D(U)$.

Consider the class of **semi-linear systems** having non-divergence form

$$\mathbf{u}_t + \mathbf{B}(\mathbf{u})\mathbf{u}_x = 0$$

with $\mathbf{B} \in \mathbb{R}^{m \times m}$. Such a system is **strictly hyperbolic** if the eigenvalues of $\mathbf{B}(z)$ are real and distinct for each $z \in \mathbb{R}^m$. For $k \in [m]$ we consider the nonzero eigenvector $\mathbf{l}_k(z)$ for the matrix $\mathbf{B}(z)^\top$, corresponding to the eigenvalue $\lambda_k(z)$, so that

$$\mathbf{B}(z)^\top \mathbf{l}_k(z) = \lambda_k(z) \mathbf{l}_k(z),$$

where $\{\mathbf{l}_k(z)\}_{k=1}^m$ are the **left eigenvectors** and $\{\mathbf{r}_k(z)\}_{k=1}^\infty$ the **right eigenvectors**.

Theorem 11.1 (1). *For $\mathbf{u}$ a smooth solution of the strictly hyperbolic system above and Φ a smooth diffeomorphism with inverse Ψ , the function $\tilde{\mathbf{u}} = \Phi(\mathbf{u})$ solves the strictly hyperbolic system*

$$\tilde{\mathbf{u}}_t + \tilde{\mathbf{B}}(\tilde{\mathbf{u}})\tilde{\mathbf{u}}_x = 0 \text{ in } \mathbb{R} \times (0, \infty)$$

for

$$\tilde{\mathbf{B}}(\tilde{z}) = D\Phi(\Psi(\tilde{z}))\mathbf{B}(\Phi(\tilde{z}))D\Phi(\tilde{z}).$$

Theorem 11.2 (2). *For the matrix function $\mathbf{B}$ smooth and hyperbolic, the eigenvalues $\lambda_k(z)$ depend smoothly on z , and one may choose $\mathbf{r}_k(z)$ and $\mathbf{l}_k(z)$ to depend smoothly on z with*

$$|\mathbf{r}_k(z)|, |\mathbf{l}_k(z)| = 1.$$

11.1 Riemann Problem

For a fixed state $z \in \mathbb{R}^n$, the k^{th} **rarefaction curve** $R_k(z_0)$ is the path of $\mathbb{R}^m$ of the solution of the ODE

$$\dot{\mathbf{v}}(s) = \mathbf{r}_k(\mathbf{v}(s))$$

which passes through z_0 .

The pair $(\lambda_k(z), \mathbf{R}_k(z))$ is **genuinely nonlinear** if

$$D\lambda_k(z) \cdot \mathbf{r}_k(z) \neq 0,$$

and is **linearly degenerate** if

$$D\lambda_k(z) \cdot \mathbf{r}_k(z) = 0$$

for all $z \in \mathbb{R}^m$.

Theorem 11.3 (1). *For some $k \in [m]$ with $(\lambda_k, \mathbf{r}_k)$ genuinely nonlinear and $u_r \in R_k^+(u_l)$, there exists a continuous integral solution $\mathbf{u}$ of Riemann's problem which is also a k -simple wave constant along lines through the origin.*

For a fixed state $z_0 \in \mathbb{R}^m$, the **shock set** is given by

$$S(z_0) = \{z \in \mathbb{R}^m \mid \mathbf{F}(z) - \mathbf{F}(z_0) = \sigma(z - z_0) \text{ for a constant } \sigma = \sigma(z, z_0)\}.$$

Theorem 11.4 (3). *For $k \in [m]$ with $(\lambda_k, \mathbf{r}_k)$ linearly degenerate,*

$$R_k(z_0) = S_k(z_0) \quad \text{and} \quad \sigma(z, z_0) = \lambda_k(z) = \lambda_k(z_0) \quad \forall z \in S_k(z_0)$$

for each $z_0 \in \mathbb{R}^m$.

11.2 Systems of Two Conservation Laws

A $w^i : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an i^{th} -**Riemann invariant** if

$$Dw^i(z) \text{ is parallel to } \mathbf{l}_j(z)$$

for all $z \in \mathbb{R}^2$, $i \neq j$.

Two smooth functions $\Phi, \Psi : \mathbb{R}^m \rightarrow \mathbb{R}$ comprise an **entropy/entropy-flux pair** for the conservation law $\mathbf{u}_t + \mathbf{F}(\mathbf{u})_x = 0$ if Φ is convex and

$$D\Phi(z)D\mathbf{F}(z) = D\Phi(z).$$

An integral solution $\mathbf{u}$ is an **entropy solution** of

$$\begin{cases} \mathbf{u}_t + \mathbf{F}(\mathbf{u})_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ \mathbf{u} = \mathbf{g} & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

if

$$\int_0^\infty \int_{-\infty}^\infty \Phi(\mathbf{u})v_t + \phi(\mathbf{u})v_x' dx dt \geq 0, \quad \text{for each } v \in C_c^\infty(\mathbb{R} \times (0, \infty)), v \geq 0$$

for each entropy/entropy-flux pair (Φ, Ψ) .

Two smooth functions $\Phi, \Psi : \mathbb{R} \rightarrow \mathbb{R}$ comprise an entropy/entropy-flux pair for the conservation law $u_t + F(u)_x = 0$ if Φ is convex and $\Phi'(z)F'(z) = \Psi'(z)$ for all $z \in \mathbb{R}$.