Primes

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Ingham

$$f(x) = O(g(x)) \iff |f(x)| \le \alpha |g(x)|,\ x > n, \alpha \in \mathbb{R}$$

$$f(x) = o(g(x)) \iff \frac{f(x)}{g(x)} \to 0$$

$$f(x) \asymp g(x) \iff f(x) = O(g(x)) \text{ and } g(x) = O(f(x))$$

$$f(x) \sim g(x) \iff \frac{f(x)}{g(x)} \to 1$$

1 Elementary

Theorem 1.1. *The series and product*

$$S(\infty) = \sum_p \frac{1}{p} \qquad \text{and} \qquad P(\infty) = \prod_p \frac{1}{1-\frac{1}{p}}$$

are both divergent.

We use

$$\frac{1}{1-u} > \frac{1-u^{m+1}}{1-u} = \sum_{k=0}^{m} u^k, \quad u \in (0,1)$$

to establish

$$P(x) > \prod_{p \le x} \left(\sum_{i=k}^{m} \frac{1}{p^k} \right),$$

and that

$$P(x) > \sum_{n=1}^{\lfloor x \rfloor} \frac{1}{n} > \int_1^{\lfloor x \rfloor + 1} \frac{1}{u}\, du > \log x$$

as well as

$$-\log(1-u) - u < \frac{1}{2} \cdot \frac{u^2}{1-u}, \quad u \in (0,1)$$

to say

$$\log P(x) - S(x) < \sum_{p \le x} \frac{1}{p^2} \cdot \frac{1}{2} \cdot \frac{1}{1-\frac{1}{p}} < \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{1}{n(n-1)} = \frac{1}{2} \quad \text{and} \quad S(x) > \log\log\left(x - \frac{1}{2}\right).$$

Theorem 1.2 (4)**.**

$$\pi(x) \sim \frac{x}{\log x} \qquad \text{as } x \to \infty.$$

Definition 1.3 (Chebyshev)**.**

$$\vartheta(x) = \sum_{p \le x} \log p, \qquad \psi(x) = \sum_{p^m \le x} \log p \qquad (x > 0).$$

Theorem 1.4.

$$\limsup_{x\to\infty} \frac{\pi(x)}{x \frac{1}{\log x}} = \limsup_{x\to\infty} \frac{\vartheta(x)}{x} = \limsup_{x\to\infty} \frac{\psi(x)}{x}.$$

Definition 1.5 (Riemann)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-\frac{1}{p^s}} \qquad (s > 1).$$

Lemma 1.6 (von Mangoldt)**.**

$$\log\zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_{p,m} \frac{1}{mp^{ms}}, \qquad -\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\frac{1}{p^s}\log p}{1-\frac{1}{p^s}} = \sum_{p,m} \frac{\log p}{p^{ms}},$$

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n), \qquad \text{and} \qquad \frac{\log p}{p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \qquad s > 1.$$

The last equation is the **generating function**. $\psi(x)$ is thence the sum of the coefficients c_n in the expansion of $-\frac{\zeta'(s)}{\zeta(s)}$ Dirichlet's series $\sum c_n \frac{1}{n^s}$.

Theorem 1.7 (via Abel)**.**

$$\log \zeta(s) = s \int_1^\infty \frac{\Pi(x)}{x^{s+1}}\, dx \qquad (s > 1).$$

We start from

$$\Pi(x) = \sum_{p^m \leq x} \frac{1}{m} = \sum_{k=0}^{\infty} \frac{1}{k} \pi\left(x^{\frac{1}{k}}\right)$$

and

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \frac{\psi(x)}{x^{s+1}}\, dx \qquad (s > 1).$$

Lemma 1.8.

$$m! = \prod_{p \leq m} p^{\sum_{k=1}^{\infty} \left\lfloor \left(\frac{m}{p}\right)^k \right\rfloor} \qquad \textit{and} \qquad \sum_{n=1}^{\infty} \frac{\log n}{n^s} = \zeta'(s).$$

Theorem 1.9.

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1), \qquad \sum_{p \leq x} \frac{1}{p} = \log\log x + \beta + O\left(\frac{1}{\log x}\right), \qquad \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}.$$

Theorem 1.10 (8)**.**

$$\zeta(\sigma + ti) = \sum_{i=1}^{\infty} \frac{1}{n^s} = \frac{s}{s-1} - s \int_i^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}$$

is an analytic continuation over the half-plane $\sigma > 0$ as a single-valued function having as its only singularity in this half-plane a simple pole at $s = 1$ with residue 1.

Theorem 1.11 (9)**.** *For $\delta \in (0, 1)$,*

$$\begin{aligned} &1) \quad |\zeta(s)| < A \log t \qquad (\sigma \geq 1,\ t \geq 2),\\ &2) \quad |\zeta'(s)| < B \log^2 t \quad (\sigma \geq 1,\ t \geq 2),\\ &3) \quad |\zeta(s)| \leq C_\delta t^{1-\delta} \qquad (\sigma \geq \delta,\ t \geq 1). \end{aligned}$$

Theorem 1.12 (10)**.**

$$\frac{1}{\zeta(s)} \in O(\log^\alpha t)$$

uniformly for $\sigma \geq 1$ as $t \to \infty$.

Theorem 1.13 (11)**.**

$$\psi_1(x) \sim \frac{1}{2} x^2$$

as $x \to \infty$.

Definition 1.14.

$$\psi_1(x) = \int_0^x \psi(u)\, du = \int_1^x \psi(u)\, du = \sum_{n \leq x} (x - n)\Lambda(n)$$

Theorem 1.15 (B)**.**

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{y^s\,ds}{s(s+1)\cdots(s+k)} = \begin{cases} 0 & (y \le 1), \\ \frac{1}{k!}\left(1-\frac{1}{y}\right)^k & (y \ge 1). \end{cases}$$

Theorem 1.16 (14)**.**

$$\zeta(\mathbf{1}-s) = \frac{2}{(2\pi)^s}\ \cos\left(\frac{\pi s}{2}\Gamma(s)\zeta(s)\right).$$

Theorem 1.17 (15)**.**

$$\xi(s) = \frac{s\pi^{-\frac{1}{2}s}}{2}(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

has

$$\xi(\mathbf{1}-s) = \xi(s), \qquad \xi(0) = \xi(1) = \frac{1}{2}, \qquad \textit{and is real for } \sigma = \frac{1}{2}.$$

Theorem 1.18 (Prime Number)**.**

$$\pi(x) \sim \frac{x}{\log x}, \quad \textit{i.e.,}\ p_n \sim n\log n$$

as $x \to \infty$*.*

Theorem 1.19 (Functional Equation)**.**

$$\zeta(s) = 2\,(2\pi)^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s),$$

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Theorem 1.20. *For* $f(z)$ *regular in* $|z-z_0| \le R$ *with at least* n *zeros in* $|z-z_0| \le r\ (< R)$*,*

$$\left(\frac{R}{r}\right)^n \le \max_{|x-z_0|=R}\frac{|f(x)|}{|f(z_0)|},$$

if $f(z_0) \neq 0$*.*

Theorem 1.21. *For*

$$f(z) = \sum_0^\infty c_n(z-z_0)^n$$

regular in $|z-z_0| < R$ *with* $\mathrm{Re}[f(z)] \le M$*,*

$$|c_n| \le \frac{2(M-\mathrm{Re}[c_0])}{R^n} \qquad \forall n,$$

and whenever $|z-z_0| \le r < R$*,*

$$|f(z)-f(z_0)| \le \frac{2r}{R-r}\left(M-\mathrm{Re}[f(z_0)]\right), \qquad \textit{and} \qquad \left|\frac{f^{(v)}(z)}{v!}\right| \le \frac{2R}{(R-r)^{v+1}}\left(M-\mathrm{Re}[f(z_0)]\right)$$

for all v*.*

The **exponent of convergence** τ of a sequence $\{|a_n|\}_i$ is given by

$$\tau = \inf\left\{\alpha \mid \sum \frac{1}{|a_n|^\alpha} < \infty\right\}.$$

Theorem 1.22. *The order ω of an integral function*

$$G(z) = e^{H(z)} \prod_n \left[\left(1 - \frac{z}{a_n}\right) \exp\left\{\sum_{m=0}^{k} \frac{1}{m}\left(\frac{z}{a_n}\right)^m\right\}\right]$$

defined by

$$\omega = \limsup_{r\to\infty} \frac{\log\log\max_{|z|=r}|G(z)|}{\log r}$$

has

$$\omega = \max\{\tau, \deg H(z)\}.$$

Further, if $\tau <$ deg $H(z)$, or $\sum \frac{1}{|a_n|^\tau}$ converges,

$$\log \sup_{|z|=r} |G(z)| \in \Theta\left(r^{\max\{\tau,h\}}\right)$$

as $r \to \infty$.

Theorem 1.23 (17)**.**

$$\log\left(\limsup_{|s|=r} |\xi(s)|\right) \sim \frac{r\log r}{2}$$

as $r \to \infty$.

Theorem 1.24 (18)**.** *$\xi(s)$ has infinitely many zeros ρ. $\sum_\rho |\rho|^{-\alpha}$ is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$. The Weierstrass product for $\xi(s)$ has form*

$$\xi(s) = e^{m+ns}\prod_\rho\left\{\left(1 - \frac{s}{\rho}\exp\left\{\frac{s}{\rho}\right\}\right)\right\}.$$

Theorem 1.25 (19)**.** *$\zeta(s)$ has no zeroes in*

$$\sigma > 1 - \frac{a}{\log(|t|+2)}$$

for some $a \in \mathbb{R}_+$.

Theorem 1.26 (20)**.** *For $\zeta(s)$ having no zeros in the domain $\sigma > 1 - \eta(|t|)$, $\eta(t)$ decreasing and of continuous derivative,*

1) $0 < \eta(t) \leq \frac{1}{2}$,
2) $\eta'(t) \to 0$ *as $t \to \infty$, and*
3) $\frac{1}{\eta(t)} \in \Theta(\log t)$ *as $t \to \infty$,*

we have

$$\frac{\zeta'}{\zeta}(s) \in O\left(\log^2|t|\right) \text{ uniformly in } \sigma \geq 1 - \beta\eta(|t|)$$

as $|t| \to \infty$, $\alpha \in (0,1)$.

Theorem 1.27 (22). *For $\eta(t)$ as above,*

$$\psi(x) = x + O\left(x \exp\left\{-\frac{\beta}{2}\varphi(x)\right\}\right),$$

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp\left\{-\frac{\beta}{2}\varphi(x)\right\}\right),$$

with

$$\varphi(x) = \min_{t \geq \mathbf{1}} \left(\eta(t) \log x + \log t\right)$$

as $x \to \infty$.

Theorem 1.28 (24).

$$\varphi(x) \in O\left(\sqrt{\log x \log\log x}\right).$$

2 Explicit Formulae

Theorem 2.1 (25). *The number of zeros of $\zeta(s)$ in the rectangle $[0,1] \times [0,K]$, $Z(K)$, has*

$$Z(K) = \mathcal{K} \log \mathcal{K} - \mathcal{K} + O(\log K)$$

for $\mathcal{K} = \frac{K}{2\pi}$ as $K \to \infty$.

Theorem 2.2 (26). *There exists a sequence of numbers $\{T_n\}$ with $n < T_n < n+1$ and*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \in O\left(\log^2 T_n\right)$$

for $\sigma \in [-1, 2]$.

Theorem 2.3 (27). *Excepting for s in the balls of half radius centered at every trivial zero of ζ,*

$$\left|\frac{\zeta'}{\zeta}(s)\right| \in O\left(\log(|s| + 1)\right).$$

Theorem 2.4 (G). *For $c > 0$, $y > 0$*

$$\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty t} \frac{y^s}{s}\, ds = \begin{cases} 0 & (y < 1), \\ \frac{1}{2} & (y = 1), \\ 1 & (y > 1). \end{cases}$$

Theorem 2.5 (30). *For the upper bound η of the real parts of zeros of $\zeta(s)$,*

$$\begin{array}{rr} 1) & \psi_1(x) = \frac{1}{2}x^2 + O\left(x^{\eta+1}\right) \\ 2) & \psi(x) = x + O\left(x^{\eta} \log^2 x\right), \\ 3) & \pi(x) = \mathrm{li}(x) + O\left(x^{\eta} \log x\right). \end{array}$$

Theorem 2.6 (31).

$$\eta = \inf\left\{\alpha \mid \psi(x) - x = O(x^{\alpha})\right\}.$$

3 Irregularities

Definition 3.1 (Oscillation)**.**

$$f(x) = \Omega_{\pm}(x) \iff 0 < \limsup_{x\to\infty} \frac{f(x) - x}{\sqrt{x} \log\log\log x} < 0$$

Theorem 3.2 (34)**.**

$$\psi(x) - x = \Omega_{\pm}\left(\sqrt{x}\log\log\log x\right).$$

as $x \to \infty$.

References

A. E. Ingham. *The Distribution of Prime Numbers.* Number 30 in Cambridge Tracts in Mathematics and Mathematical Physics. Cambridge University Press, Cambridge, 1932.